Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 6900–6929
Journal of Differential Equations
www.elsevier.com/locate/jde

Electrodynamic two-body problem for prescribed initial data on a straight line

D.-A. Deckert, G. Hinrichs *

Mathematisches Institut der LMU, Theresienstr. 39, 80333 München, Germany

Received 17 July 2015; revised 30 December 2015
Available online 29 January 2016

Abstract

Electrodynamic interaction between point charges can be described by a system of ODEs involving advanced and retarded delays – the so-called Fokker–Schwarzschild–Tetrode (FST) equations. In special situations, approximate equations can be derived which are purely retarded. Upon omission of the terms describing radiation friction, these are called Synge equations. In both cases, few mathematical results are available on existence and uniqueness of solutions. We investigate the situation of two like point-charges in $3+1$ space–time dimensions restricted to motion on a straight line. We give a priori estimates on the asymptotic motion and, using a Leray–Schauder argument, prove: 1) Existence of solutions to the FST equations on the future or past half-line given finite trajectory segments; 2) Global existence of the Synge equations for Cauchy data; 3) Global existence of a FST toy model. Furthermore, we give a sufficient criterion that uniquely distinguishes solutions by means of finite trajectory segments.

MSC: 34K99; 83A05

Keywords: Fokker–Schwarzschild–Tetrode electrodynamics; Wheeler–Feynman electrodynamics; Synge equations; Delay differential equations

* Corresponding author.
E-mail addresses: deckert@math.ucdavis.edu (D.-A. Deckert), hinrichs@math.lmu.de (G. Hinrichs).

http://dx.doi.org/10.1016/j.jde.2016.01.015

1. Introduction and main results

The direct electrodynamic interaction between point-charges is a prime example for systems of ordinary differential equations involving retarded and advanced 'delays'. Its fundamental equations of motion can be inferred by means of an informal variational principle of the action S which is given as a functional of the world-lines of the point-charges, $z_i : \mathbb{R} \to \mathbb{R}^4$, $\tau \mapsto z_i(\tau)$:

$$S[z_1, \dots, z_n] = -\sum_{i=1}^{N} \int m_i \sqrt{dz_{i\mu}(\tau) dz_i^\mu(\tau)} \\ - \sum_{\substack{(i,j)\in\{1,\dots,N\}^2 \\ i\neq j}} \frac{e_i e_j}{2} \int dz_{i\mu} \int dz_j^\mu \delta\Big((z_i(\tau) - z_j(\sigma))^2\Big). \tag{1}$$

In this relativistic notation, τ is the parametrization of the world-line, and space–time point $z_i = (z_i^\mu)_{\mu=0,1,2,3}$ denotes the time and space coordinates, z_i^0 and $\mathbf{z} = (z_i^j)_{j=1,2,3}$, respectively. The integral $\int dz_i^\mu$ is to be interpreted as the line integral $\int d\tau \dot{z}_i^\mu(\tau)$, where dots denote derivatives w.r.t. τ. Furthermore, for space–time points a, b, we use the summation convention $a_\mu b^\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$ to denote the *indefinite Minkowski scalar product* and employ the abbreviation $(a - b)^2 = (a_\mu - b_\mu)(a^\mu - b^\mu)$, which is referred to as the squared *Minkowski distance* between points a and b. The symbol δ denotes the one-dimensional Dirac delta distribution, m_i denotes the mass of the particles, e_i the respective charge, and we chose units such that the speed of light and the vacuum permittivity equal one. The integral in the first summand in (1) measures the arc length of the i-th world line using the Minkowski metric, and the double integral in the second summand gives rise to an interaction between pairs of world lines whenever the Minkowski distance between z_i and z_j is zero. The extrema of the action S, i.e., z_i such that $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} S[z_i + \epsilon\delta z_i] = 0$, fulfill the Fokker–Schwarzschild–Tetrode (FST) equations (also known as Wheeler–Feynman equations):

$$m_i \ddot{z}_i^\mu(\tau) = e_i \sum_{j=1, j\neq i}^{N} \frac{1}{2} \sum_{\pm} F_{j\pm}^{\mu\nu}\big(z_i(\tau)\big) \dot{z}_{i\nu}(\tau), \qquad i = 1, 2, \dots, N \tag{2}$$

with the electromagnetic field tensors $F_{j\pm}^{\mu\nu}(x) = \partial/\partial x_\mu A_j^\nu(x) - \partial/\partial x_\nu A_j^\mu(x)$ given by means of the four-vector potentials

$$A_{j\pm}^\mu(x) = e_j \frac{\dot{z}_{j\pm}}{(x - z_{j\pm})_\mu \dot{z}_{j\pm}^\mu}, \qquad z_{j\pm}^\mu = z_j^\mu(\tau_{j\pm}), \qquad x^0 - z_j^0(\tau_{j\pm}) = \pm|\mathbf{x} - \mathbf{z}_j(\tau_{j\pm})|. \tag{3}$$

Equation (2) is a special relativistic form of Newton's law of motion, in electrodynamics referred to as Lorentz equation. The field tensors $F_{j+}^{\mu\nu}$, $F_{j-}^{\mu\nu}$ are the so-called advanced (+) and retarded (−) electrodynamic Liénard–Wiechert fields [11] which are generated by the j-th charge, respectively. They are given in terms of the corresponding potentials A_{j+}^μ, A_{j-}^μ, which are functionals of the world line $\tau \mapsto z_j(\tau)$ since the parameters $\tau_{j\pm}$ are defined implicitly as solutions to the last equation in line (3). This implicit equation is due to the delta function in (1) and has a nice geometrical interpretation. When evaluating $F_{j\pm}^{\mu\nu}(x)$ at $x = z_i$ as in the Lorentz law of motion (2),