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On the regularizing effect for unbounded solutions of first-order Hamilton–Jacobi equations ☆

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Abstract

We give a simplified proof of regularizing effects for first-order Hamilton–Jacobi Equations of the form $u_t + H(x, t, Du) = 0$ in $\mathbb{R}^N \times (0, +\infty)$ in the case where the idea is to first estimate u_t. As a consequence, we have a Lipschitz regularity in space and time for coercive Hamiltonians and, for hypo-elliptic Hamiltonians, we also have an Hölder regularizing effect in space following a result of L.C. Evans and M.R. James.

MSC: 35F21; 35D35; 35D40

Keywords: First-order Hamilton–Jacobi Equations; Viscosity solutions; Regularizing effects

1. Introduction

In this short paper we give a new proof of regularizing effects for Hamilton–Jacobi Equations

$$u_t + H(x, t, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, \infty)\,, \tag{1.1}$$

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in the case when the aim is to estimate u_t first. This new proof is inspired by ideas introduced in [3] and then simplified in [2]; a precise comparison between the results and ideas of [3,2] and ours is provided just after the statement of the main results of this article, at the end of Section 2. More classical proofs can be found in [1] but with stronger assumptions and more tedious proofs.

The model equations we have in mind are

$$u_t + |A(x,t)Du|^m = f(x,t) \quad \text{in } \mathbb{R}^N \times (0,\infty)\,, \tag{1.2}$$

where f is a continuous (typically bounded from below) function and A takes values in the set of $N \times N$ symmetric matrices. For such equations, we consider two cases: the *coercive case* for which A is invertible and, as a consequence $|A(x,t)Du|^m \to +\infty$ as $|p| \to +\infty$, and the *non-coercive case* where A may be degenerate. In both cases, we provide regularizing effects for *bounded from below solutions*. The main improvement in the assumptions is easy to describe in the coercive case since we just require that A, f are continuous in x (no uniform continuity assumptions) and, in particular, f may have some growth at infinity. In the non-coercive case, analogous results hold except that we have to impose far more restrictive assumptions on the t-dependence of the equation.

Of course, for (1.2), the equation implies that $u_t \le f(x,t)$ in $\mathbb{R}^N \times (0,\infty)$ and therefore we just need an estimate from below for u_t.

To do so, our approach consists in using the exponential transform, $v := -\exp(-u)$. Notice that provided u is bounded below (then we can always assume that u is nonnegative), we get that v is bounded since $-1 \le v \le 0$. Moreover, v solves a new Hamilton–Jacobi equation

$$v_t + G(x,t,v,Dv) = 0\,, \quad \text{with } G(x,t,v,p) := -vH\Big(x,t,-\frac{p}{v}\Big)\,. \tag{1.3}$$

In order to estimate v_t, a key property (as in all the regularizing effects proofs) is to have a large enough, positive G_v when $v_t = -G$ is large (but negative) i.e. when G is large and positive. This leads to an assumption on $(H_p \cdot p - H)(x,t,p)$ which is classical except that, here, this quantity has to be large when $H(x,t,p)$ is large, and not when $|p|$ is large as it is classical for the estimate on Du.

The paper is organized as follows: in Section 2, we state our main result which provides a short time regularizing effect in time both in the cases of coercive and non-coercive Hamiltonians. Then we deduce full regularizing effects, i.e. globally in space and time. The proofs of the main theorems are given in Section 3. Then, in Section 4, we treat several explicit examples. We have put some technical results about Hamiltonian G in an appendix.

2. Assumptions and main results

2.1. Assumptions

In order to state and prove our results, we use several structure conditions, which all rely on the following fundamental hypothesis:

(**H0**) *The function H is continuous in $\mathbb{R}^N \times [0,T] \times \mathbb{R}^N$ and there exists $c_0 = c_0(H) \ge 0$ such that*

- *H is locally Lipschitz in the p-variable, in a neighborhood of the set $\{(x,t,p);\ H(x,t,p) \ge c_0\}$;*