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Non-local effects by homogenization or 3D–1D dimension reduction in elastic materials reinforced by stiff fibers

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Abstract

We first consider an elastic thin heterogeneous cylinder of radius of order ε: the interior of the cylinder is occupied by a stiff material (fiber) that is surrounded by a soft material (matrix). By assuming that the elasticity tensor of the fiber does not scale with ε and that of the matrix scales with ε^2, we prove that the one dimensional model is a nonlocal system.

We then consider a reference configuration domain filled out by periodically distributed rods similar to those described above. We prove that the homogenized model is a second order nonlocal problem.

In particular, we show that the homogenization problem is directly connected to the 3D–1D dimensional reduction problem.

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1. Introduction

Quite often nature combines two, or more, materials with complementary properties to generate a structural material whose performance and functionality supersede those of monolithic materials, [26]. Natural materials usually present a complex hierarchical structure, with characteristic dimensions spanning from the nanoscale to the macroscale, that is quite difficult to replicate in fabricated materials, [27]. Nowadays, engineers and material scientists design and fabricate novel materials, inspired by natural materials, with a complex, but not hierarchical, structure. Two materials with completely different properties are appropriately arranged to create advanced functional materials that are lightweight, with remarkable strength, stiffness, and toughness, [3,22,23]. The material properties of a two component material can be tuned by appropriately choosing the individual components, their morphology, their size, and arrangement, [3]. The understanding of the mechanical response of these advanced functional materials is an issue of paramount importance for industrial applications.

In the present paper, we study structures made up by a stiff and a soft material; the latter usually provides energy dissipation, toughness, ductility, and makes the structure lighter. This combination is quite common in engineering and biological composites.

We first derive, within the framework of linear elasticity, the elastic problem governing the motion of a rod composed of a stiff and a soft material. This is achieved by considering a sequence of problems posed on cylindrical reference configurations of diameters that scale with a parameter ε, and by taking a variational limit as ε approaches zero. The interior of the cylindrical regions is occupied by the stiff material (fiber), while the surrounding part by a soft material (matrix), whose modelling is achieved by scaling the elasticity tensor by ε^2. In this way the ratio between the components of the elasticity tensor of the fiber and those of the matrix is $1/\varepsilon^2$. No assumption on the material symmetry of the body is made and the material could be inhomogeneous also within each of the stiff and the soft regions.

We show that the limit problem may be written equivalently as two independent systems: the first posed on the matrix, while the second takes into account the elastic energy of only the fiber and the loads applied to the entire body. Mathematically speaking, the problem posed on the matrix is local, while the problem posed on the fiber is non-local since also the loads applied outside of the fiber region enter into the problem. For any distribution of loads the problem is still non-local, since in order to get the global information about the limit displacement the problems posed on the matrix and on the fibers have to be both solved. Leaving technicalities aside, the problem on the fiber determines a Bernoulli–Navier type of displacement, while the problem on the matrix determines the deviation from the Bernoulli–Navier type of displacement that takes place within the matrix region. The need of a displacement correction within the matrix region could be explained by the fact that the matrix is much more deformable than the fiber.

As a first step, to find the variational limit, we deduced a priori bounds on the displacements. These are easily obtained within the fiber region by means of Korn inequality, while a more intricate argument is needed within the matrix region. Indeed, since there is a loss of uniform ellipticity within the matrix region, only through a judicious application of several Poincaré and Korn inequalities, that also exploit the a priori bounds obtained within the fiber region, we are able to prove a priori bounds within the matrix region.

We also address a related homogenization problem. We consider a reference configuration domain Ω filled out by periodically distributed rods similar to those considered in the rod problem mentioned above. To completely fill Ω, we take rods with a square base of size ε. The body is therefore made of two regions: one occupied by a stiff material (fibers), while the second is