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J. Differential Equations 260 (2016) 2060–2090

Journal of Differential Equations

www.elsevier.com/locate/jde

Pacemakers in large arrays of oscillators with nonlocal coupling ☆

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Received 5 September 2014; revised 24 September 2015

Available online 23 October 2015

Abstract

We model pacemaker effects of an algebraically localized heterogeneity in a 1 dimensional array of oscillators with nonlocal coupling. We assume the oscillators obey simple phase dynamics and that the array is large enough so that it can be approximated by a continuous nonlocal evolution equation. We concentrate on the case of heterogeneities with positive average and show that steady solutions to the nonlocal problem exist. In particular, we show that these heterogeneities act as a wave source. This effect is not possible in 3 dimensional systems, such as the complex Ginzburg–Landau equation, where the wavenumber of weak sources decays at infinity. To obtain our results we use a series of isomorphisms to relate the nonlocal problem to the viscous eikonal equation. We then use Fredholm properties of the Laplace operator in Kondratiev spaces to obtain solutions to the eikonal equation, and by extension to the nonlocal problem.

Keywords: Pacemakers; Nonlocal coupling; Kondratiev spaces

1. Introduction

The collective behavior in systems of coupled oscillators has attracted a tremendous amount of interest. Self-organized synchronization in large systems appears particularly dramatic when

☆ GJ and AS acknowledge support by the National Science Foundation through grants DMS-0806614 and DMS-1311740.

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http://dx.doi.org/10.1016/j.jde.2015.09.054

0022-0396/

coupling effects are seemingly weak [22]. A substantial part of the work has been devoted to the study of such collections of oscillators in the strikingly simple and explicit Kuramoto model [9]. Synchronization and desynchronization as well as a plethora of more complex states have been found, and boundaries (or phase transitions) have been characterized [21,1]. On the other hand, it is well known that the collective behavior may well depend on the type of coupling, as well as the type of internal dynamics at nodes. Of particular interest have been spiky oscillators in neuroscience with their quite different phase response curves, or phase-amplitude descriptions near Hopf bifurcations.

Our interest here focuses modestly on a rigorous description of pacemakers. Roughly speaking, we ask if and how a small collection of oscillators can influence the collective behavior of a large ensemble. This question has been addressed in numerous contexts. One observed dramatic influence manifests itself through the occurrence of target patterns. Phenomenologically, a faster (or slower) patch of oscillators entrains neighbors and a phase-lag gradient propagates through the medium according to an eikonal equation.

The analysis of such phenomena is notoriously complicated by the absence of spectral gaps in the linearization at the synchronized state, inherently related to the presence of a neutral phase in the medium. Standard perturbation analysis in a large collection of oscillators, based on an Implicit Function Theorem, is valid only for extremely small sizes of perturbations and fails to capture key phenomena. In an infinite medium, the range of the linearization is not closed, so that simple matched asymptotics cannot be justified. In fact, in an infinite medium (and also, with some corrections, in large media), one observes that the system relaxes to a frequency-synchronized state, but the collective frequency depends in unusual ways on the perturbation parameter. Characterizing, for instance, the deviation of the localized patch of oscillators from the ensemble background by ε, the collective frequency will change with $\omega \sim \varepsilon^2$ for $\varepsilon > 0$ and remain constant for $\varepsilon < 0$, in a one-dimensional medium. It is this general phenomenon that we are concerned with in this paper.

One can ask questions of perturbative nature in many different circumstances. First, one can consider various types of oscillators, ranging from simple phase oscillators $\phi' = \omega$, over gauge-invariant phase-amplitude oscillators, $A' = (1 + \mathrm{i}\omega)A - (1 + \mathrm{i}\gamma)A|A|^2$, to general asymptotically stable periodic orbits $u_*(-\omega t)$ in an ODE $u' = f(u)$. On the other hand, one can look at simple scalar diffusive coupling, or, most generally, dynamics on networks. Our focus is on *simple* internal phase dynamics, but non-local coupling along a line. Previous results have studied phase-dynamics, formally derived from the complex Ginzburg–Landau equation for amplitude-phase oscillators [20], and general stable periodic orbits with diffusive coupling, but in a one-dimensional context [16] or with radial symmetry [7]. Radial symmetry was removed as an assumption in [5] in the complex Ginzburg–Landau equation in 3 dimensions.

Phase dynamics can be derived and shown to approximate dynamics on long temporal and spatial scales [2]. A general form of the dynamics is

$$\phi_t = d\Delta\phi - \kappa|\nabla\phi|^2 + \omega_*, \qquad \phi \in \mathbb{R}/(2\pi\mathbb{Z}). \tag{1.1}$$

Substituting $\phi \to \phi + \omega_* t$, thus exploiting the phase invariance, we can assume that $\omega_* = 0$. The solutions $\phi \equiv \bar{\phi}$ correspond to the spatially synchronized state. One can show that this synchronized state is asymptotically stable under localized (L^1) perturbations, with decay rate given by the “effective viscosity” $\sim dt^{-n/2}$ [4].