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Convergence to equilibrium for solutions of an abstract wave equation with general damping function

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Abstract

We prove convergence to a stationary solution as time goes to infinity of solutions to abstract nonlinear wave equation with general damping term and gradient nonlinearity, provided the trajectory is precompact. The energy is supposed to satisfy a Kurdyka–Łojasiewicz gradient inequality. Our aim is to formulate conditions on the function g as general as possible when the damping is a scalar multiple of the velocity, and this scalar depends on the norm of the velocity, $g(|u_t|)u_t$. These turn out to be estimates and a coupling condition with the energy but not global monotonicity. When the damping is more general, we need an angle condition.

Keywords: Abstract wave equation with damping; Convergence to equilibrium; Łojasiewicz inequality

1. Introduction

This work has been inspired by a result of Chergui presented in [4], where the following semilinear damped wave equation

$$u_{tt}(t,x) + |u_t(t,x)|^\alpha u_t(t,x) = \Delta u + f(x, u(t,x)), \quad t \geq 0, \; x \in \Omega \subset \mathbb{R}^N, \tag{1}$$

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with zero boundary conditions on $\partial\Omega$ is studied. It is proved that every bounded solution has relatively compact trajectory and that for certain values of α every solution with relatively compact trajectory converges to a stationary solution. The set of admissible α's depends on the Łojasiewicz exponent of the operator $\Delta + f(x,\cdot)$.

The main goal of this paper is to study the above equation with a more general damping term $g(|u_t|)u_t$ resp. $G(u,u_t)$ instead of $|u_t|^\alpha u_t$ and to obtain convergence to equilibrium for solutions with relatively compact trajectory. We prove our result in a more general setting assuming an abstract gradient operator $E'(u)$ instead of $\Delta u + f(x,u)$. Thus, we study the equation

$$u_{tt}(t) + g(|u_t(t)|)u_t(t) = E'(u(t)), \quad t > 0, \tag{2}$$

where the damping is $g(|u_t|)u_t$, u_t is the velocity and g is a scalar function. Our analysis shows also the way how to generalise the result to a more general model, an anisotropic, inhomogeneous medium where the damping need not point into the direction of the velocity, that is the equation

$$u_{tt}(t) + G(u(t),u_t(t)) = E'(u(t)), \quad t > 0. \tag{3}$$

In this formulation we obtain a generalisation of [1, Theorem 4] for the ordinary differential equation

$$\ddot{u}(t) + G(u(t),\dot{u}(t)) = E'(u(t)), \quad t > 0, \tag{4}$$

for $u : [0,+\infty) \to \mathbb{R}^N$, for more general damping than in [1]; see also [3].

We denote $V := H_0^1(\Omega)$, $H := L^2(\Omega)$, $V^* := H^{-1}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is open and bounded. Let $E \in C^2(V)$ and let $g : [0,+\infty) \to [0,+\infty)$ be given. Consider the problem (2) with given initial values $u(0) = u_0 \in V$, $\quad u_t(0) = u_1 \in H$. Let us assume that there exists a solution $u \in C^1([0,+\infty),H) \cap C([0,+\infty),V)$ such that $|u_t|^2 g(|u_t|) \in L^1((0,+\infty),L^1(\Omega))$ and assume that the trajectory $\{(u(t),u_t(t)) : t \geq 0\}$ is relatively compact in $V \times H$. Then there exists a sequence $t_n \to +\infty$ such that $(u(t_n),u_t(t_n))$ converges to some $(\varphi,\psi) \in V \times H$ and one can show that $\psi = 0$ (see [4]). The question we are interested in is whether

$$\lim_{t\to+\infty} u(t) = \varphi \ ?$$

In [4, Theorem 1.4] Chergui gave a positive answer to this question for the equation (1) under suitable assumptions on f provided α satisfies the following two conditions:

1. $0 < \alpha < \frac{\theta}{1-\theta}$, where θ is a Łojasiewicz exponent depending on E, $0 < \theta \leq \frac{1}{2}$,
2. $\alpha < \frac{4}{N-2}$.

The first condition says that the damping term $g(|u_t|)u_t$ is not too small near zero (which seems to be a reasonable condition). It also estimates the growth at infinity but it can be seen from the proof that one does not need this estimate. The second condition says that the growth of g at $+\infty$ is not too fast and it stems from a Sobolev embedding needed in the proof. It also means that the growth of g at zero is not too small, but we show that this estimate at zero is not necessary. From the physical interpretation we would say that the bigger is the damping term, the better is the convergence or the stabilisation effect.