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Algorithmic framework for group analysis of differential equations and its application to generalized Zakharov–Kuznetsov equations

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Abstract

In this paper, we explain in more details the modern treatment of the problem of group classification of (systems of) partial differential equations (PDEs) from the algorithmic point of view. More precisely, we revise the classical Lie algorithm of construction of symmetries of differential equations, describe the group classification algorithm and discuss the process of reduction of (systems of) PDEs to (systems of) equations with smaller number of independent variables in order to construct invariant solutions. The group classification algorithm and reduction process are illustrated by the example of the generalized Zakharov–Kuznetsov (GZK) equations of form $u_t + (F(u))_{xxx} + (G(u))_{xyy} + (H(u))_x = 0$. As a result, a complete group classification of the GZK equations is performed and a number of new interesting nonlinear invariant models which have non-trivial invariance algebras are obtained. Lie symmetry reductions and exact solutions for two important invariant models, i.e., the classical and modified Zakharov–Kuznetsov equations, are constructed. The algorithmic framework for group analysis of differential equations presented in this paper can also be applied to other nonlinear PDEs.

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1. Introduction

One of the most famous two-dimensional generalizations (together with Kadomtsev–Petviashvili equation) of the Korteweg–de Vries (KdV) equation is given by the Zakharov–Kuznetsov (ZK) equation

$$u_t + au_{xxx} + bu_{xyy} + cuu_x = 0. \tag{1}$$

It was first derived by Zakharov and Kuznetsov [41] to describe nonlinear ion-acoustic waves in a magnetized plasma. More precisely, they considered a plasma in a strong magnetic field, $\mathbf{B} = \boldsymbol{B}\hat{z}$, with cold ions and hot electrons ($T_e \gg T_i$). The ion motions are described by the following equations

$$n_t + \nabla \cdot (n\mathbf{u}) = 0, \quad \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{e}{m_i}\nabla\phi + \mathbf{u} \times \Omega_i,$$
$$\nabla^2\phi = -4\pi e(n - n_e),$$

where n is number density of ions, $\mathbf{u} = (u, v, w)$ is ion velocity, m_i is ion mass, ϕ is electric potential, $\Omega_i = \frac{e\mathbf{B}}{m_i c}$ is scaled magnetic field and $n_e = \exp(\frac{e\phi}{KT_e})$.

After introducing the dimensionless variables and approximating the x-component of $\mathbf{u}$ by the polarization shift these equations look like

$$n_t - (n\phi_{tx})_x + (nw)_z = 0, \quad w_t = \phi_{tx}w_x + ww_z - \phi_z,$$
$$\alpha\phi_{xx} + \phi_{zz} = e^\phi - n.$$

Now, after a change of independent variables $\xi = \varepsilon^{1/2}(z - t)$, $\eta = \varepsilon^{1/2}x$, $\tau = \varepsilon^{3/2}t$ assuming a solution of the latter equations of the form

$$n = 1 + \sum_{j=1}^{\infty} \varepsilon^j n_j, \quad \phi = \sum_{j=1}^{\infty} \varepsilon^j \phi_j, \quad w = \sum_{j=1}^{\infty} \varepsilon^j w_j,$$

one gets that $n_1 = \phi_1 = w_1$ with ϕ_1 being solution of

$$\phi_{1,t} + \phi_1\phi_{1,\xi} + \frac{1}{2}(\phi_{1,\xi\xi\xi} + (1+\alpha)\phi_{1,\xi\eta\eta}) = 0$$

which has the form (1).

In the more realistic situation in which the electrons are non-isothermal, Munro and Parkes [25,26] showed that, with an appropriate modified form of the electron number density proposed by Schamel [33], a reductive perturbation procedure leads to a modified form of the Zakharov–Kuznetsov (mZK1) equation, namely