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Existence and regularity of solution for a stochastic Cahn–Hilliard/Allen–Cahn equation with unbounded noise diffusion

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Abstract

The Cahn–Hilliard/Allen–Cahn equation with noise is a simplified mean field model of stochastic microscopic dynamics associated with adsorption and desorption-spin flip mechanisms in the context of surface processes. For such an equation we consider a multiplicative space-time white noise with diffusion coefficient of linear growth. Applying techniques from semigroup theory, we prove local existence and uniqueness in dimensions $d = 1, 2, 3$. Moreover, when the diffusion coefficient satisfies a sub-linear growth condition of order α bounded by $\frac{1}{3}$, which is the inverse of the polynomial order of the nonlinearity used, we prove for $d = 1$ global existence of solution. Path regularity of stochastic solution, depending on that of the initial condition, is obtained a.s. up to the explosion time. The path regularity is identical to that proved for the stochastic Cahn–Hilliard equation in the case of bounded noise diffusion. Our results are also valid for the stochastic Cahn–Hilliard equation with unbounded noise diffusion, for which previous results were established only in the framework of a bounded diffusion coefficient.

As expected from the theory of parabolic operators in the sense of Petrovskĭı, the bi-Laplacian operator seems to be dominant in the combined model.

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1. Introduction

1.1. The stochastic equation

We consider the Cahn–Hilliard/Allen–Cahn equation with multiplicative space-time noise:

$$\begin{cases} u_t = -\varrho\Delta\Big(\Delta u - f(u)\Big) + \Big(\Delta u - f(u)\Big) + \sigma(u)\dot{W} \quad \text{in} \quad \mathcal{D}\times[0,T), \\ u(x,0) = u_0(x) \quad \text{in} \quad \mathcal{D}, \\ \frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{D}\times[0,T). \end{cases} \tag{1.1}$$

Here, $\mathcal{D}$ is a rectangular domain in $\mathbb{R}^d$ with $d = 1, 2, 3$, $\varrho > 0$ is a "physical diffusion" constant, f is a polynomial of degree 3 with a positive leading coefficient, such as $f = F'$ where $F(u) = (1-u^2)^2$ is a double equal-well potential. The "noise diffusion" coefficient $\sigma(\cdot)$ is a Lipschitz function with sub-linear growth, $\dot{W}$ is a space-time white noise in the sense of Walsh [20], and ν is the outward normal vector. In addition, we assume that the initial condition u_0 is sufficiently integrable or regular, depending on the desired results on the solution. Obviously, when $\sigma := 1$, the noise in (1.1) becomes additive.

In this paper, as in [3], we will analyze the more general case of multiplicative noise. However, unlike [3], we consider a more general Lipschitz coefficient σ with sub-linear growth such that

$$|\sigma(u)| \leq C(1 + |u|^\alpha), \tag{1.2}$$

for some $\alpha \in (0, 1]$ and a positive constant C.

In the sequel, we will give sufficient conditions on the initial condition u_0 so that:

(1) a unique local maximal solution exists when $d = 1, 2, 3$, for $\alpha = 1$, that is when σ satisfies the classical linear growth condition;
(2) when $\alpha < \frac{1}{3}$, i.e. when α is strictly smaller than the inverse of the polynomial order of the nonlinear function f, a global solution exists with Lipschitz path-regularity for $d = 1$.

The stochastic Cahn–Hilliard equation can be considered as a special case of our model. Therefore, when the function σ satisfies the aforementioned sub-linear growth assumption, our method extends all the results of [3] on existence and uniqueness of a local maximal solution when $d = 1, 2, 3$, and on global existence and path-regularity, when $d = 1$, for the solution of the stochastic Cahn–Hilliard equation with a multiplicative noise; in reference [3] C. Cardon-Weber considered a bounded diffusion coefficient. It seems to us that there is a gap in the proof of global existence given in reference [3], on page 793. Indeed the various constraints imposed on the parameters d, a, q, r and γ' lead to a contradiction; however we did not disprove the statement of the corresponding Theorem 1.3. The argument we use in this paper to prove global existence is different from that in [3] and is based on the Gagliardo Nirenberg inequality. Using the factorization method for the stochastic term, we derive a path regularity similar to that obtained in [3].