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The sixth Painlevé transcendent and uniformization of algebraic curves ☆

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Abstract

We exhibit a remarkable connection between sixth equation of Painlevé list and infinite families of explicitly uniformizable algebraic curves. Fuchsian equations, congruences for group transformations, differential calculus of functions and differentials on corresponding Riemann surfaces, Abelian integrals, analytic connections (generalizations of Chazy's equations), and other attributes of uniformization can be obtained for these curves. As byproducts of the theory, we establish relations between Picard–Hitchin's curves, hyperelliptic curves, punctured tori, Heun's equations, and the famous differential equation which Apéry used to prove the irrationality of Riemann's $\zeta(3)$.

Keywords: Painlevé-6 equation; Picard–Hitchin solutions; Algebraic curves; Theta-functions; Automorphic functions; Fuchsian equations

Contents

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1. Introduction

The first example of general solution to the famous sixth Painlevé transcendent

$$\mathcal{P}_6\colon\quad y_{xx}=\frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-x}\right)y_x^2-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{y-x}\right)y_x$$
$$+\frac{y(y-1)(y-x)}{x^2(x-1)^2}\left\{\alpha-\beta\frac{x}{y^2}+\gamma\frac{x-1}{(y-1)^2}-\left(\delta-\frac{1}{2}\right)\frac{x(x-1)}{(y-x)^2}\right\} \qquad (1)$$

was obtained in 1889 before equation (1) itself had been derived by Richard Fuchs in 1905 [33]. This case corresponds to parameters $\alpha=\beta=\gamma=\delta=0$ and is referred frequently to as Picard's solution [63]. Surprisingly, but the second one was obtained by N. Hitchin [40] after more than one hundred years. It corresponds to parameters $\alpha=\beta=\gamma=\delta=\frac{1}{8}$. Presently these solutions are the only instances, up to automorphisms in the space $(\alpha,\beta,\gamma,\delta)$, when solution of (1) is known in its full generality.

One year after Fuchs, Painlevé [61] gave a remarkable form to (1) which is known nowadays as the $\wp$-form of $\mathcal{P}_6$-equation. The modern representation of this result is given by the nice