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On the convergence of the two-dimensional second grade fluid model to the Navier–Stokes equation

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Abstract

We consider the equations governing the motion of incompressible second grade fluids in a bounded two-dimensional domain with Navier-slip boundary conditions. We first prove that the corresponding solutions are uniformly bounded with respect to the normal stress modulus α in the L^∞-H^1 and the L^2-H^2 time–space norms. Next, we study their asymptotic behavior when α tends to zero, prove that they converge to regular solutions of the Navier–Stokes equations and give the rate of convergence in terms of α.

MSC: 35Q30; 76A05; 76B03

Keywords: Second grade fluids; Navier–Stokes equations; Navier-slip boundary conditions; Uniform a priori estimates; Rate of convergence

1. Introduction

The second grade fluid model forms a subclass of differential type fluids of complexity 2, and is one of the simplest constitutive models for flows of non-Newtonian fluids that can predict normal stress differences (cf. [23] or [21]). The Cauchy stress tensor $\mathbb{T}$ for a homogeneous incompressible second grade fluid is given by a constitutive equation of the form

$$\mathbb{T} = -p\mathbb{I} + \nu A_1(\boldsymbol{u}) + \alpha_1 A_2(\boldsymbol{u}) + \alpha_2 \left(A_1(\boldsymbol{u})\right)^2, \tag{1.1}$$

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where p denotes the hydrodynamic pressure, ν is the viscosity of the fluid, α_1 and α_2 are viscoelastic parameters (normal stress moduli), $\boldsymbol{u}$ is the velocity field, A_1 and A_2 are the first two Rivlin–Ericksen tensors defined by

$$A_1(\boldsymbol{u}) = 2D\boldsymbol{u}, \qquad A_2(\boldsymbol{u}) = \left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right) A_1(\boldsymbol{u}) + A_1(\boldsymbol{u})\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top A_1(\boldsymbol{u})$$

with $D\boldsymbol{u} = \frac{\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^\top}{2}$ standing for the symmetric part of the velocity gradient. According to [8], if the fluid modeled by equation (1.1) is to be compatible with thermodynamics, then

$$\nu \geq 0, \qquad \alpha_1 \geq 0, \qquad \alpha_1 + \alpha_2 = 0. \tag{1.2}$$

We refer to [9] for a critical and extensive historical review of second-order (and higher order) fluid models and, in particular, for a discussion on the sign of the normal stress moduli. Here we will restraint to the simplified case $\alpha_1 + \alpha_2 = 0$, with $\alpha_1 \geq 0$ and $\nu > 0$, and supplement the obtained governing equation by Navier-slip boundary conditions. More precisely, setting $\alpha = \alpha_1$, we will consider the following problem

$$\begin{cases} \frac{\partial}{\partial t}(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}) - \nu\Delta\boldsymbol{u} + \mathbf{curl}\,(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}) \times \boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } Q, \\ \operatorname{div}\boldsymbol{u} = 0 & \text{in } Q, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad (\boldsymbol{n}\cdot D\boldsymbol{u})\cdot\boldsymbol{\tau} = 0 & \text{on } \Sigma, \\ \boldsymbol{u}(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.3}$$

where f is the given body force, u_0 is the initial data, $Q =]0,T[\times\Omega$, $\Sigma =]0,T[\times\Gamma$, T is a fixed positive number, $\Omega \subset \mathbb{R}^2$ is a bounded domain with boundary Γ, $n = (n_1, n_2)$ and $\tau = (-n_2, n_1)$ are the unit normal and tangent vectors to the boundary Γ. As this equation is set in dimension two, the vector $\boldsymbol{u}$ is written in the form $\boldsymbol{u} = (u \equiv (u_1, u_2), 0)$ in order to define the vector product and the curl

$$\mathbf{curl}\,\boldsymbol{u} = (0, 0, \operatorname{curl} u) \qquad \text{with } \operatorname{curl} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

Even for this simple but mathematically interesting model, the problem is still difficult since the nonlinear term involves derivatives with higher order than the ones appearing in the viscous term. In the inviscid case ($\nu = 0$), the second-grade fluid equations are called α-Euler equations. Initially proposed as a regularization of the incompressible Euler equations, they are geometrically significant and have been interpreted as a model of turbulence (cf. [13] and [14]). They also inspired another variant, called the α-Navier–Stokes equations that turned out to be very relevant in turbulence modeling (cf. [11,10] and the references therein). These equations contain the regularizing term $-\nu\Delta(\boldsymbol{u} - \alpha\Delta\boldsymbol{u})$ instead of $-\nu\Delta\boldsymbol{u}$, making the dissipation stronger and the problem much easier to solve than in the case of second-grade fluids. When $\alpha = 0$, the α-Navier–Stokes and the second grade fluid equations are equivalent to the Navier–Stokes equations.

The case of Dirichlet boundary conditions has received a lot of attention. It was studied for the first time in [22] and [5] for both steady and unsteady cases. A Galerkin's method in the basis of the eigenfunctions of the operator $\mathbf{curl}(\mathbf{curl}(\boldsymbol{u} - \alpha\Delta\boldsymbol{u}))$ was especially designed to decompose the problem into a mixed parabolic–hyperbolic type, looking for the velocity $\boldsymbol{u}$ as a solution of a Stokes-like system coupled to a transport equation satisfied by $\mathbf{curl}\,(\boldsymbol{u} - \alpha\Delta\boldsymbol{u})$. Under minimal restrictions on the data, this approach allows the authors to establish the existence of solutions