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Infinite families of harmonic self-maps of spheres

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Abstract

For each of the spheres $\mathbb{S}^n$, $n \geq 5$, we construct a new infinite family of harmonic self-maps, and prove that their members have Brouwer degree ± 1 or ± 3. These self-maps are obtained by solving a singular boundary value problem. As an application we show that for each of the special orthogonal groups SO(4), SO(5), SO(6) and SO(7) there exist two infinite families of harmonic self-maps.

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1. Introduction

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds and U be a domain of M with piecewise C^1 boundary. The energy functional of φ over U is given by

$$E_U(\varphi) = \int_U |d\varphi|^2 \omega_g.$$

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A smooth map $f : M \to N$ is called harmonic if it is a critical point of the energy functional. For the special case $M = N = \mathbb{S}^n$, where $\mathbb{S}^n$ is equipped with the standard metric, the Euler–Lagrange equations of the energy functional are given by the elliptic system

$$\Delta f + |df|^2 f = 0,$$

where Δ denotes the Laplace–Beltrami operator for the sphere $\mathbb{S}^n$. Finding solutions of this partial differential equation is difficult in general. By imposing symmetry conditions on the solution one can sometimes reduce this problem to finding solutions of an ordinary differential equation.

In this paper we restrict ourselves to self-maps of spheres which are equivariant with respect to the cohomogeneity one action

$$\mathrm{SO}(m_0+1) \times \mathrm{SO}(m_1+1) \times \mathbb{S}^{m_0+m_1+1} \to \mathbb{S}^{m_0+m_1+1}, \qquad (A, B, v) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} v.$$

In this case the Euler Lagrange equations reduce to the singular ordinary differential equation

$$\ddot{r}(t) = ((m_1 - m_0)\csc 2t - (m_0 + m_1)\cot 2t)\,\dot{r}(t) - m_1 \frac{\sin 2r(t)}{2\cos^2 t} + m_0 \frac{\sin 2r(t)}{2\sin^2 t}.$$

It was shown in [15] that each solution of this ordinary differential equation which satisfies $r(0) = 0$ and $r(\frac{\pi}{2}) = (2\ell + 1)\frac{\pi}{2}$, $\ell \in \mathbb{Z}$, yields a harmonic self-map of $\mathbb{S}^{m_0+m_1+1}$. The above ordinary differential equation and boundary value problem are henceforth referred to as (m_0, m_1)-ODE and (m_0, m_1)-BVP, respectively.

The goal of this paper is the construction of solutions of the (m_0, m_1)-BVP and the examination of their properties.

Initial value problem. In order to find solutions of the (m_0, m_1)-BVP we use a shooting method at the degenerate point $t = 0$. This is possible since for each $v \in \mathbb{R}$ there exists a unique solution r_v of the (m_0, m_1)-ODE with $r(0) = 0$ and $\dot{r}(0) = v$. This initial value problem is solved in Section 3.

The cases $2 \le m_0 \le 5$. We show that for $2 \le m_0 \le 5$ there exist infinitely many solutions of the (m_0, m_1)-BVP. These solutions are labeled by the number of intersections of r and $\frac{\pi}{2}$, the so-called *nodal number*.

Theorem A. *Let* $2 \le m_0 \le 5$ *and* $m_0 \le m_1$. *For each* $k \in \mathbb{N}$ *there exists a solution of the* (m_0, m_1)*-BVP with nodal number* k.

For the special case that the multiplicities coincide, reflecting a solution of the (m, m)-BVP on the point $(\frac{\pi}{4}, \frac{\pi}{4})$ yields again a solution of the (m, m)-BVP. We use this fact to show that for $2 \le m \le 5$ there exist infinitely many solutions of the (m, m)-BVP with nodal number 0.

Theorem B. *If* $m_0 = m_1 =: m$ *and* $2 \le m \le 5$ *there exists a countably infinite family of solutions of the* (m, m)*-BVP with nodal number* 0.

Theorems A and B are proved in Section 4 and Section 6, respectively.

The cases $m_0 \ge 6$. We explain why for $m_0 \ge 6$ a construction analogous to that for the cases $2 \le m_0 \le 5$ is not possible. The reason is simply that for $m_0 \ge 6$ the nodal number is bounded from above.