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Optimal bilinear control of Gross–Pitaevskii equations with Coulombian potentials ☆

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Abstract

In this paper, we consider an optimal bilinear control problem for the Gross–Pitaevskii equations with Coulombian potentials. We show the well-posedness of the problem and the existence of an optimal control. In addition, the first order optimality system is rigorously derived. In particular, we prove the Fréchet-differentiability of the unconstrained functional. We extend the study of Hintermüller et al. (2013) [15] to more general power nonlinearities and unbounded potentials.

MSC: 35Q55; 49J20

Keywords: Optimal bilinear control problem; Gross–Pitaevskii equations; Optimal condition; Fréchet-differentiability

1. Introduction

The mathematical research for optimal bilinear control of systems governed by partial differential equations has a long history, see [12,24] for a general overview. Especially, quantum control problems described by Schrödinger equations have received a lot of attention in the past

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decades, see [6,16,17]. From the mathematical point of view, quantum control problems are a specific example of optimal control problems [11], which consist in minimizing a cost functional depending on the solution of the state equation and in characterizing the minimum of the functional by an optimality condition. Recently, control problems have been extensively studied for linear Schrödinger equations in [2–4,7,18,20,22,26,27,32], for nonlinear Schrödinger equations in [5,13,15,23,21,28–31].

In [15], Markowich et al. considered the following nonlinear Schrödinger equation of Gross–Pitaevskii type:

$$\begin{cases} iu_t = -\frac{1}{2}\Delta u + U(x)u + \phi(t)V(x)u + \lambda|u|^{2\sigma}u, \ (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\lambda \geq 0$, i.e., the case of defocusing nonlinearity, $U(x)$ is a subquadratic potential, consequently restricting initial data $u_0 \in \Sigma := \{u \in H^1(\mathbb{R}^d), \text{ and } xu \in L^2(\mathbb{R}^d)\}$. The authors presented a novel choice for the cost term, which is different from that in [1–3,18], and based on the corresponding physical work performed throughout the control process. Firstly, they showed well-posedness of the problem and existence of an optimal control under the assumption $V \in W^{1,\infty}(\mathbb{R}^d)$. Secondly, they proved only the Gâteaux-differentiability of unconstrained functional $\mathcal{F}$. The reason is that they obtained only Lipschitz continuity of the solution $u(\phi)$ with respect to control ϕ for each fixed direction. Thirdly, the first order optimality system was rigorously derived under the assumptions $\sigma \in \mathbb{N}$ with $\sigma < \frac{2}{d-2}$ and $V \in W^{m,\infty}(\mathbb{R}^d)$ for some $m > d/2$. These assumptions imply $\sigma = 1$ in the case of $d = 3$. Finally, they discussed some possible generalizations and presented two problems. One is how to provide a rigorous mathematical framework for control potential $V(x)$ which is unbounded with respect to $x \in \mathbb{R}^d$. The other is how to extend these results (with some technical effort) to the case of focusing nonlinearities, i.e., $\lambda < 0$, provided $\sigma < 2/d$. Notice that there is no σ such that $\sigma \in \mathbb{N}$ and $\sigma < 2/d$ in the case of $d = 2, 3$. Therefore, for solving this problem, one must find a way to cancel the assumption $\sigma \in \mathbb{N}$.

In this paper, we consider the optimal bilinear control problem for the following Gross–Pitaevskii equation:

$$\begin{cases} iu_t = -\frac{1}{2}\Delta u + U(x)u + \phi(t)\frac{1}{|x|}u + \lambda|u|^{2\sigma}u, \ (t,x) \in [0,\infty) \times \mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

where $u(t,x)$ is a complex-valued function in $(t,x) \in [0,\infty) \times \mathbb{R}^3$, $u_0 \in \Sigma$, $\phi(t)$ denotes the control parameter, $U(x)$ is subquadratic in the sense of [14], that is:

$$U \in C^\infty(\mathbb{R}^3;\mathbb{R}), \quad \text{with } \partial^\alpha U \in L^\infty(\mathbb{R}^3), \quad \forall\, \alpha \in \mathbb{N}^3, \ |\alpha| \geq 2.$$

In particular, we give answers to the problems raised by Markowich et al., and extend their study to several directions:

1. Our results hold for the Coulombian potential $\frac{1}{|x|}$. For physical reasons we shall only consider the three-dimensional case here, that is the spatial variable x is assumed to be in $\mathbb{R}^3$. In fact, we could deal with more general unbounded potentials in $\mathbb{R}^d$. Our proofs remain valid provided that the potential $V : \mathbb{R}^d \to \mathbb{R}$ in (1.1) satisfies: