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Existence and decay estimates of solutions to complex Ginzburg–Landau type equations

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Abstract

This paper deals with the initial-boundary value problem (denoted by (CGL)) for the complex Ginzburg–Landau type equation $\frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-1}u - \gamma u = 0$ with initial data $u_0 \in L^p(\Omega)$ in the case $1 < q < 1 + 2p/N$, where Ω is bounded or unbounded in $\mathbb{R}^N$, $\lambda > 0$, $\alpha, \beta, \gamma, \kappa \in \mathbb{R}$. There are a lot of studies on local and global existence of solutions to (CGL) including the physically relevant case $q = 3$ and $\kappa > 0$. This paper gives existence results with precise properties of solutions and rigorous proof from a mathematical point of view. The physically relevant case can be considered as a special case of the results. Moreover, in the case $\kappa < 0$, local and global existence of solutions with the decay estimate $\|u(t)\|_{L^p(\Omega)} \le c_1 e^{-c_2 t}$ (c_1, c_2 are positive constants) is obtained under some conditions. The key to the local existence is to construct a semigroup $\{e^{t[(\lambda+i\alpha)\Delta]}\}$ and its L^p-L^q estimate. On the other hand, the key to the global existence is to derive estimates for solutions by using a kind of interpolation inequality with $\mathrm{Re}\langle |v|^{p-2}v, -(\lambda + i\alpha)\Delta v\rangle$.

MSC: primary 35Q56; secondary 35A01, 47D06

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1. Introduction and results

Let Ω be a bounded or unbounded domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$ with compact C^2-boundary, $\Omega = \mathbb{R}^N_+$ or $\Omega = \mathbb{R}^N$. In this paper we consider the following initial-boundary value problem for the complex Ginzburg–Landau type equation in $L^p(\Omega)$ $(1 < p < \infty)$:

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{q-1}u - \gamma u = 0 & \text{in } \Omega \times (0, \infty), \\ u = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(\cdot, 0) = u_0 & \text{on } \Omega, \end{cases} \tag{CGL}$$

where u is a complex-valued unknown function, $i = \sqrt{-1}$, $\alpha, \beta, \gamma, \kappa \in \mathbb{R}$, $\lambda > 0$, $q > 1$. We call the equation in (CGL) the complex Ginzburg–Landau *type* equation, because (CGL) is mostly studied in the case $q = 3$ and $\kappa > 0$ to be explained later.

The complex Ginzburg–Landau equation describes spatial pattern formation or the onset instabilities in non-equilibrium fluid dynamical systems; in particular, the case $q = 3$ and $\kappa > 0$ is the most physically relevant (see Aranson and Kramer [1]). In this context there are a lot of studies on existence and behavior of solutions to the equation. For example, the behavior was established by Doering, Gibbon, Holm, and Nicolaenko [9,10]; turbulence in the equation was investigated by Bartuccelli, Constantin, Doering, Gibbon, and Gisselfalt [3,4]; existence and behavior of weak and strong solutions to the generalized equation were established by Doering, Gibbon, and Levermore [11] and Bartuccelli, Gibbon, and Oliver [5]; shaper bounds and attractors on unbounded domains were obtained by Mielke [19–21]; an approach to the equation as perturbations of nonlinear Schrödinger equations was given by Cruz-Pacheco, Levermore and Luce [8].

In the case that the nonlinear term has the opposite sign ($\kappa < 0$), which corresponds to a subcritical (backward or inverse) bifurcation, higher-order nonlinear terms are usually essential (see [1, Sec. VI.A]). The subcritical case is indeed of interest, and it is known that there are interesting bounded solutions. There is an extensive literature on this, that goes back at least as far at the paper [30] by Thual and Fauve. The motivation for this and much subsequent interest in the subcritical case has been that it arises frequently in applications, e.g., binary fluid convection and shear flow instability being perhaps the most prominent, partly because of many experiments.

From a purely mathematical point of view, existence and uniqueness of solutions to (CGL) and related problems were also studied by Yang [31], Levermore and Oliver [15], Ginibre and Velo [13,14], Okazawa and Yokota [24,25], Okazawa [23], Yokota and Okazawa [33], Matsumoto and Tanaka [17,18], Yokota [32], Clément, Okazawa, Sobajima, and Yokota [7] and Lu, Bates, Lü, and Zhang [16]. We would like to focus on C^1-in-time solutions to (CGL) as in [23] and [18] (see Definition 1.1 below).

Local existence of C^1-in-time solutions to (CGL) was obtained by [23, Proposition 1.1] when Ω is bounded, $\kappa \in \mathbb{R}$, $1 \le p < \infty$, $1 < q \le 1 + \frac{2p}{N}$ and $u_0 \in L^p(\Omega)$. However, there seem to be only few studies on the local existence when Ω is an unbounded general domain because the semigroup $\{e^{t[(\lambda+i\alpha)\Delta]}\}$ has not been sufficiently studied yet, e.g., its L^p-L^q estimate. In [19–21] the behavior of solutions was studied even if Ω is unbounded by using the weighted L^p-norms $\|u\|_{p,\rho} = \left(\int_\Omega \rho(x)|u(x)|^p\,dx\right)^{1/p}$, where $\rho : \mathbb{R}^N \to (0, \infty)$ is a suitable weight with $|\nabla\rho(x)| \le \rho_0\rho(x)$ and $\int_{\mathbb{R}^N} \rho(x)\,dx < \infty$. In this paper we show that (CGL) has a local C^1-in-time solution with $\kappa \in \mathbb{R}$ when Ω is unbounded by using the usual L^p-norm.