Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 259 (2015) 423–453
Journal of Differential Equations
www.elsevier.com/locate/jde

# On a reaction–diffusion equation with Robin and free boundary conditions ☆

Xiaowei Liu [a,b], Bendong Lou [b,*]

[a] College of Science, Qilu University of Technology, Jinan 250353, China
[b] Department of Mathematics, Tongji University, Shanghai 200092, China

Received 25 September 2014; revised 14 January 2015
Available online 20 February 2015

**Abstract**

This paper studies the following problem

$$\begin{cases} u_t = u_{xx} + f(u), & 0 < x < h(t),\ t > 0, \\ u(0,t) = b u_x(0,t), & t > 0, \\ u(h(t),t) = 0, \quad h'(t) = -u_x(h(t),t), & t > 0, \\ h(0) = h_0, \quad u(x,0) = \sigma\phi(x), & 0 \leqslant x \leqslant h_0 \end{cases}$$

where $f$ is an unbalanced bistable nonlinearity, $b \in [0,\infty)$, $\sigma \geqslant 0$ and $\phi$ is a compactly supported $C^2$ function. We prove that, there exists $\sigma^* > 0$ such that, vanishing happens when $\sigma < \sigma^*$ (i.e., $h(t) < M$ for some $M > 0$ and $u(\cdot,t)$ converges as $t \to \infty$ to 0 uniformly in $[0, h(t)]$); spreading happens when $\sigma > \sigma^*$ (i.e., $h(t) - c^*t$ tends to a constant for some $c^* > 0$, $u(\cdot,t)$ converges to a positive stationary solution locally uniformly in $[0,\infty)$ and to a traveling semi-wave with speed $c^*$ near $x = h(t)$); in the transition case when $\sigma = \sigma^*$, $\|u(\cdot,t) - V(\cdot - \xi(t))\|_{H^2([0,h(t)])}$ tends to 0 as $t \to \infty$, where $\xi(t)$ is a maximum point of $u(\cdot,t)$ and $V$ is the unique even positive solution of $V'' + f(V) = 0$ subject to $V(\infty) = 0$. Moreover, with respect to $b$ and $f$, $\xi(t) = P \ln t + Q + o(1)$ for some $P > 0$ and $Q \in \mathbb{R}$, or, $\xi(t) \to z$ for some root $z$ of $V(-z) = bV'(-z)$.


*MSC:* 35K57; 35B40; 35R35; 35B20

☆ This research was partly supported by NSFC (No. 11271285).
* Corresponding author.
*E-mail address:* blou@tongji.edu.cn (B. Lou).

http://dx.doi.org/10.1016/j.jde.2015.02.012
0022-0396/

*Keywords:* Reaction–diffusion equation; Long time behavior; Free boundary condition; Robin boundary condition; Sharp threshold

## 1. Introduction

Consider the following free boundary problem

$$\begin{cases} u_t = u_{xx} + f(u), & 0 < x < h(t),\ t > 0, \\ u(0,t) = bu_x(0,t), & t > 0, \\ u(h(t),t) = 0, \quad h'(t) = -u_x(h(t),t), & t > 0, \\ h(0) = h_0, \quad u(x,0) = u_0(x), & 0 \leqslant x \leqslant h_0 \end{cases} \tag{1}$$

where $b \in [0,\infty)$, $x = h(t)$ is a moving boundary to be determined together with $u(x,t)$ and $f$ is an **unbalanced bistable** nonlinearity satisfying

$$\begin{cases} f \in C^1([0,\infty)),\ f(0) = 0 > f'(0),\ f(\cdot) < 0 \text{ in } (0,\alpha), \\ f(\cdot) > 0 \text{ in } (\alpha,1),\ f(\cdot) < 0 \text{ in } (1,\infty),\ \inf_{s>0} f(s)/s > -\infty, \\ \text{for } F(u) := -2\int_0^u f(s)ds,\ F(\theta) = 0 \text{ for some } \theta \in (\alpha,1). \end{cases} \tag{\textbf{F}}$$

The initial function $u_0$ is chosen from $\mathscr{X}(h_0)$, where, for some $h_0 > 0$,

$$\mathscr{X}(h_0) := \left\{ \phi \,\middle|\, \begin{array}{l} \phi \in C^2([0,h_0]),\ \phi(0) = b\phi'(0), \\ \phi \geqslant, \not\equiv 0 \text{ in } [0,h_0],\ \phi \equiv 0 \text{ in } [h_0,\infty) \end{array} \right\}.$$

Problem (1) may be used to describe the spreading of a new or invasive biological/chemical species, with the free boundary $h(t)$ representing the spreading front of the species whose density is denoted by $u(x,t)$. The (Stefan) free boundary condition indicates that the front invades at a rate that is proportional to the magnitude of the spatial population gradient there. Such a boundary condition was used for population models in [6,12,13] etc., for tumor, protocell and wound healing models in [10,22–24], etc. It can be derived from Fick's law of diffusion by considering the "population loss" based on Allee effect at the front (cf. [6] for details).

Problem (1) with Neumann boundary condition $u_x(0,t) = 0$ was recently studied by Du and Lin [12] for logistic nonlinearity $f(u) = u(1-u)$, and by Du and Lou [13] for general $f$. Among others they obtained a dichotomy result for the solutions of (1) with a monostable $f$: for any positive solution, either spreading happens (i.e., $\lim_{t\to\infty} h(t) = \infty$ and $\lim_{t\to\infty} u(\cdot,t) = 1$); or vanishing happens (i.e., $h(t) < M$ for some $M > 0$ and $\lim_{t\to\infty} u(\cdot,t) = 0$). The vanishing phenomenon is a remarkable result which reveals the difference between the free boundary problem and the Cauchy problem (in the latter case, it is known that the problem with a monostable $f$ has the so-called *hair trigger effect*, which implies that any positive solution will spread, cf. [4,5]). In case $f$ is a bistable nonlinearity satisfying **(F)**, [13] studied (1) with Neumann boundary condition $u_x(0,t) = 0$. They gave a rather complete description on the long time behavior of the solution $u(\cdot,t;\sigma\phi)$ of (1) with initial datum $u_0 = \sigma\phi$. They proved that there exists $\sigma^* > 0$ such that, vanishing happens for $u(\cdot,t;\sigma\phi)$ when $\sigma < \sigma^*$; spreading happens when $\sigma > \sigma^*$; and in the transition case when $\sigma = \sigma^*$, $u(\cdot,t;\sigma^*\phi)$ converges as $t \to \infty$ to $V(\cdot)$ locally uniformly in $x \in \mathbb{R}$, where $V$ is the unique even positive solution of $V'' + f(V) = 0$ subject to $V(\infty) = 0$ (i.e. the so-called *ground state*). The approach of the last result relies on the fact that $u_x(x,t) < 0$ for