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Merging homoclinic solutions due to state-dependent delay

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Abstract

For $\frac{\pi}{2} < \alpha < \frac{5\pi}{2}$ the linear differential equation

$$x'(t) = -\alpha\, x(t-1)$$

with constant time lag is unstable and hyperbolic. For α sufficiently close to $\frac{5\pi}{2}$ we construct a state-dependent delay $d_U(\phi) \in (0,2)$ on an open set in the space $C^1([-2,0])$, with $d_U(\phi) = 1$ for ϕ close to 0, so that the nonlinear equation

$$x'(t) = -\alpha\, x(t - d_U(x_t))$$

has a one-parameter-family of entire solutions $c^{[r]} : \mathbb{R} \to \mathbb{R}$, $|r| < r_U$, which are homoclinic to zero and merge in finite time at some $t_y > 0$, $c^{[r]}(t) = c^{[0]}(t)$ for $|r| < r_U$ and $t \geq t_y$. This complements earlier results on complicated solution behaviour caused by state-dependent delay and shows that there exist delay functionals for which certain transversality and regularity conditions along homoclinic solutions are violated.

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Keywords: Delay differential equation; State-dependent delay; Homoclinic solution

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1. Introduction

For autonomous ordinary differential equations $x' = f(x)$ information about solution behaviour is read off from the vectorfield, at least in low dimensions. If the system involves a delay, as for example in simple-looking equations of the form

$$x'(t) = -\mu\, x(t) + f(x(t-1))$$

with a smooth function $f : \mathbb{R} \to \mathbb{R}$ and a parameter $\mu > 0$, then the semiflow generated by the equation lives on a function space, a generating vectorfield is no longer visible from the differential equation, and information about how the data α and f shape the solution behaviour is less accessible. Nevertheless, experience and results obtained during the past decades provide us with some insight into the organisation of the state space by the semiflow, for example, how a global attractor is structured by a Morse–Smale decomposition [6], or stratified by invariant manifolds [4], and when to expect planar motion with periodic orbits, or chaos, see the recent survey [16] and comments in the monographs [2,1].

Much less is known about the effects on solution behaviour produced by a delay which is not constant but variable, and state-dependent. There are nonlinear equations of the form

$$\epsilon\, x'(t) = -x(t) + k\, x(t - 1 - c(x(t)))$$

with $\epsilon > 0$, $k > 1$, and a function $c : \mathbb{R} \to \mathbb{R}$ which have periodic solutions while for $c = 0$ the linear equation

$$\epsilon\, x'(t) = -x(t) + k\, x(t-1)$$

with constant time lag has all nontrivial entire solutions $\mathbb{R} \to \mathbb{R}$ unbounded [7,8], and the asymptotic shape of periodic solutions of the nonlinear equation, for $\epsilon \searrow 0$, depends in a subtle way on the function c. In [14] we started with an even simpler linear equation, namely,

$$x'(t) = -\alpha\, x(t-1), \tag{1}$$

which for $0 < \alpha \neq \frac{\pi}{2} + 2\pi j$, $j \in \mathbb{N}_0$ has all nontrivial entire solutions unbounded. For α in a small interval $\left(\alpha_1, \frac{5\pi}{2}\right)$ we constructed a state-dependent delay $d_{\mathcal{O}}(\phi) \in (0,2)$ on an open set $\mathcal{O}$ in the state space $C = C([-2,0])$, with

$$d_{\mathcal{O}}(\phi) = 1 \quad \text{for} \quad \phi \quad \text{close to} \quad 0 \in \mathcal{O},$$

so that the nonlinear equation

$$x'(t) = -\alpha\, x(t - d_{\mathcal{O}}(x_t))$$

has a solution $h : \mathbb{R} \to \mathbb{R}$ which is homoclinic to zero, $h \neq 0$ and

$$h(t) \to 0 \quad \text{as} \quad |t| \to \infty.$$