Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 259 (2015) 510–530
Journal of Differential Equations
www.elsevier.com/locate/jde

The Glassey conjecture for nontrapping obstacles

Chengbo Wang [1]

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Received 18 January 2014; revised 3 February 2015
Available online 26 February 2015

Abstract

We verify the 3-dimensional Glassey conjecture for exterior domain (M, g), where the metric g is asymptotically Euclidean, provided that certain local energy assumption is satisfied. The radial Glassey conjecture exterior to a ball is also verified for dimension three or higher. The local energy assumption is satisfied for many important cases, including exterior domain with nontrapping obstacles and flat metric, exterior domain with star-shaped obstacle and small asymptotically Euclidean metric, as well as the nontrapping asymptotically Euclidean manifolds $(\mathbb{R}^n, g)$.

MSC: 35L70; 35L15

Keywords: Glassey conjecture; Exterior domain; Semilinear wave equations; Local energy estimates; KSS estimates; Asymptotically Euclidean manifold

1. Introduction

The purpose of this paper is to show how integrated local energy estimates for certain linear wave equations involving asymptotically Euclidean perturbations of the standard Laplacian lead to optimal existence theorems for the corresponding small amplitude nonlinear wave equations with power nonlinearities in the derivatives. The problem is an analog of the Glassey conjecture

E-mail address: wangcbo@zju.edu.cn.
URL: http://www.math.zju.edu.cn/wang.
[1] The author was supported by Zhejiang Provincial Natural Science Foundation of China LR12A01002, the Fundamental Research Funds for the Central Universities 2012QNA3002, NSFC 11301478, 11271322 and J1210038.

http://dx.doi.org/10.1016/j.jde.2015.02.023

in the exterior domain, see Hidano–Wang–Yokoyama [11] and the references therein. In particular, for spatial dimension three, we prove global existence of small amplitude solutions for any power greater than a critical power, as well as the almost global existence for the critical power. The critical power is the same as that on the Minkowski space. On the other hand, for dimension four and higher, the current technology could only apply for the radial case, and we obtain existence results with certain lower bound of the lifespan, which is sharp in general. The non-radial case is still open, even for the Minkowski space, when the spatial dimension is four or higher.

Let us start by describing the asymptotically Euclidean manifolds (M, g), where $M = \mathbb{R}^n \backslash \mathcal{K}$ with smooth and compact obstacle $\mathcal{K}$ and $n \geq 3$. Without loss of generality, when $\mathcal{K}$ is nonempty, we assume the origin lies in the interior of $\mathcal{K}$ and $\mathcal{K} \subset B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$. By asymptotically Euclidean, we mean that

$$g = g_0 + g_1(r) + g_2(x), g = g_{ij}(x)dx^i dx^j = \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j \tag{H1}$$

where (g_{ij}) is uniformly elliptic, $(g_{0,ij}) = Diag(1, 1, \cdots, 1)$ is the standard Euclidean metric, the first perturbation g_1 is *radial*, and

$$\sum_{ijk} \sum_{l\geq 0} 2^{l(i+|\alpha|-1)} \|\nabla^\alpha g_{i,jk}\|_{L_x^\infty(A_l)} \lesssim 1, \forall\alpha. \tag{H1.1}$$

Here, $A_0 = \{|x| \leq 1\}$, $A_l = \{2^{l-1} \leq |x| \leq 2^l\}$ for $l \geq 1$, and we say g_1 is radial, if, when writing out the metric g, with $g_2 = 0$, in polar coordinates $x = r\omega$ with $r = |x|$ and $\omega \in \mathbb{S}^{n-1}$, we have

$$g = g_0 + g_1 = \tilde{g}_{11}(r)dr^2 + \tilde{g}_{22}(r)r^2 d\omega^2.$$

In this form, the assumption (H1.1) for g_1 is equivalent to the following requirement

$$\sum_{l\geq 0} 2^{|\alpha|l} \|\nabla^\alpha (\tilde{g}_{11} - 1, \tilde{g}_{22} - 1)\|_{L_x^\infty(A_l)} \lesssim 1, \forall\alpha. \tag{H1.2}$$

When $g = g_0 + \delta(g_1 + g_2)$ with sufficient small parameter δ, we call it a *small* perturbation. Notice that this sort of assumption and its role in local energy estimates seems to have started with Tataru [34] for Schrödinger equations and Metcalfe–Tataru [23] for wave equations. See also Tataru [35], Metcalfe–Tataru–Tohaneanu [24] for similar assumptions regarding the interaction with rotations.

We shall consider Dirichlet-wave equations on (M, g),

$$\begin{cases} \Box_g u \equiv (\partial_t^2 - \Delta_g)u = F, \ x \in M, t > 0 \\ u(t, x) = 0, x \in \partial M, t > 0 \\ u(0, x) = \phi(x), \partial_t u(0, x) = \psi(x), \end{cases} \tag{1.1}$$

where Δ_g is the Laplace–Beltrami operator associated with g.

Now we can state the local energy assumption that we shall make

Hypothesis 2. For any $R > 1$, we have

$$\|(\partial u, u)\|_{L_t^2 L_x^2(B_R)} \leq C(\|\phi\|_{H^1} + \|\psi\|_{L^2} + \|F\|_{L_t^2 L_x^2}), \tag{H2}$$