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Well-posedness results for abstract generalized differential equations and measure functional differential equations

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Abstract

In the first part of the paper, we consider nonlinear generalized ordinary differential equations whose solutions take values in infinite-dimensional Banach spaces, and prove new theorems concerning the existence of solutions and continuous dependence on initial values and parameters. In the second part, we apply these results in the study of nonlinear measure functional differential equations and impulsive functional differential equations with infinite delay.

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1. Introduction

The concept of a generalized ordinary differential equation was originally introduced by J. Kurzweil in [20] as a tool in the study of continuous dependence of solutions to ordinary differential equations of the usual form $x'(t) = f(x(t), t)$. He observed that instead of dealing directly with the right-hand side f, it might be advantageous to work with the function $F(x, t) = \int_{t_0}^{t} f(x, s)\,\mathrm{d}s$, i.e., the primitive to f. In this connection, he also introduced an integral

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whose special case is the Kurzweil–Henstock integral. Gauge-type integrals are well-known to specialists in integration theory, but they are also becoming more popular in the field of differential equations (see e.g. [5]).

Over the years, it became clear that the theory of generalized differential equations is not only useful in the study of classical nonautonomous differential equations (see e.g. [2]), but also represents a suitable tool for the investigation of equations with discontinuous solutions. (For other approaches to equations with discontinuous solutions or right-hand sides, such as measure differential equations, equations in Filippov's or Krasovskii's sense, distributional differential equations, or impulsive differential equations, see e.g. [6,7,14,18,19,22,25,31].) In particular, generalized differential equations encompass other types of equations, such as equations with impulses, dynamic equations on time scales, functional differential equations with impulses, or measure functional differential equations (see e.g. [10,11,13,27,32,37] and the references there). To deal with functional differential equations, it is necessary to consider generalized equations whose solutions take values in infinite-dimensional Banach spaces; this fact provides a motivation to the study of abstract generalized differential equations.

Unfortunately, the existing theory for abstract generalized equations is not as powerful as in the finite-dimensional case. The only exception is the class of linear equations, where the results are quite satisfactory (see [17,27,28]). Our goal is to rectify this situation and obtain new results concerning well-posedness of solutions to abstract nonlinear generalized differential equations under reasonably weak assumptions on the right-hand sides.

In the theory of classical ordinary differential equations, it is well-known that Picard's theorem on the local existence and uniqueness of solutions of the problem

$$x'(t) = f(x(t), t), \quad t \in [a, b], \quad x(a) = x_0, \tag{1.1}$$

can be improved in the following way: Instead of working with a locally Lipschitz continuous right-hand side, it is enough to assume that f is a continuous function satisfying

$$\|f(x, t) - f(y, t)\| \le \omega(\|x - y\|),$$

where $\omega : [0, \infty) \to [0, \infty)$ is a continuous increasing function such that $\omega(0) = 0$ and

$$\lim_{v \to 0+} \int_v^u \frac{\mathrm{d}r}{\omega(r)} = \infty \tag{1.2}$$

for every $u > 0$. This existence–uniqueness result is known as Osgood's theorem (in [29], W.F. Osgood identified (1.2) to be a sufficient condition for uniqueness). The situation in finite-dimensional spaces is rather simple, since the existence of a local solution follows from Peano's theorem, and condition (1.2) together with Bihari's inequality guarantee uniqueness.

Remarkably, Peano's theorem is no longer valid in infinite-dimensional Banach spaces, but Osgood's theorem remains true (see e.g. [8, Theorem 3.2] and [35]). In this case, condition (1.2) is necessary to prove both existence and uniqueness. The proof uses the fact that a continuous right-hand side can be uniformly approximated by locally Lipschitz continuous right-hand sides; the corresponding initial-value problems have unique solutions, which are uniformly convergent to a solution of the original equation.