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Pullback attractors for multi-valued non-compact random dynamical systems generated by reaction–diffusion equations on an unbounded domain ☆

Yejuan Wang *, Jingyu Wang

School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, PR China

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Abstract

In this paper we study pullback attractors of reaction–diffusion equations on an unbounded domain with non-autonomous deterministic as well as stochastic forcing terms for which the uniqueness of solutions need not hold. We first present the existence and structure of pullback attractors of multi-valued non-compact random dynamical systems. Then we prove the existence of pullback attractors in $L^2(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ for multi-valued non-compact random dynamical systems associated with the reaction–diffusion equations on $\mathbb{R}^n$, and the identical relation of pullback attractors in different spaces is also provided. In particular, the measurability of pullback attractors is established for reaction–diffusion equations on $\mathbb{R}^n$.

MSC: 35B40; 35B41; 37L30

Keywords: Multi-valued non-compact random dynamical system; Pullback attractor; Reaction–diffusion equation; Unbounded domain

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* Corresponding author.
E-mail address: wangyj@lzu.edu.cn (Y. Wang).

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1. Introduction

This paper is devoted to the long term behavior of the following reaction–diffusion equation defined on $\mathbb{R}^n$:

$$du + (\lambda u - \Delta u)dt = f(x,u)dt + g(t,x)dt + \sum_{j=1}^{m} h_j dw_j, \tag{1.1}$$

with initial data $u(\tau, x) = u_\tau(x)$, where $t > \tau$ with $\tau \in \mathbb{R}$, $x \in \mathbb{R}^n$, λ is a positive constant, $g \in L^2_{loc}(\mathbb{R}; L^2(\mathbb{R}^n))$, f is a continuous nonlinear function satisfying some dissipativeness and growth conditions, and $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space.

Stochastic differential equations of this type arise from many biological and physical systems, where g is a time-dependent input signal and $\{w_j\}_{j=1}^m$ are Wiener processes used to test the impact of stochastic fluctuations on g. The concept of pullback attractors for random dynamical systems was introduced in [10,14], for instance, and the existence of such attractors has been studied for stochastic PDEs with uniqueness of solutions, see, e.g., [3,4,11–13,16,19,23,24,29,30] and the references therein, but little is known for stochastic PDEs without uniqueness of solutions. Here we show the existence, regularity and measurability of pullback attractors for the stochastic reaction–diffusion equation (1.1) defined on $\mathbb{R}^n$ without uniqueness of solutions.

First, we extend the result for the existence of pullback attractors for multi-valued random dynamical systems in [6] to the multi-valued cocycles with two parametric spaces Q and Ω, where Q is responsible for the non-autonomous deterministic forcing terms and Ω is responsible for the stochastic forcing terms, and then prove a sufficient and necessary condition for existence of pullback attractors for such multi-valued cocycles. As demonstrated in [24] for the single-valued case, we show that the structure of pullback attractors for such multi-valued cocycles is determined by the complete orbits. The sufficient condition for measurability of pullback attractors for such multi-valued cocycles is also presented.

Secondly, we derive uniform estimates on the solutions when $t \to \infty$ with the purpose of proving the existence of pullback absorbing sets, and provide uniform estimates on the far-field values of solutions to overcome the difficulty caused by the unboundedness of the domain. This method was used to prove the asymptotic compactness of solutions of reaction–diffusion equations on $\mathbb{R}^n$, see, e.g., [4,20,24,29,30] and also [27] for the delay case. It is worth mentioning that, in this paper the tail estimates in a space of higher regularity are established for stochastic dissipative PDEs without uniqueness of solutions.

Thirdly, we investigate the existence of pullback attractors in $L^2(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ for Eq. (1.1) on $\mathbb{R}^n$. The existence of pullback attractors in $L^p(\mathbb{R}^n)$ is proved in [29] for a stochastic reaction–diffusion equation on $\mathbb{R}^n$ with uniqueness of solutions. Asymptotic a priori estimate method was introduced in [28] to prove asymptotic compactness for the deterministic version of (1.1) in bounded domains, and later used in several other works, see, e.g., [16,20,21,25,29]. Some ideas for checking the asymptotical upper-semicompactness of solutions in $L^p(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ are developed in this paper to circumvent the difficulty caused by the nonlinearity and the multi-valued case. The reader is also referred to [5–7,9,15,17,18,22,27] for asymptotic behavior of multi-valued dynamical systems.

Finally, we prove the measurability of pullback attractors with respect to the $\mathbb{P}$-completion of $\mathcal{F}$. The existence of random attractors for first-order stochastic lattice systems with non-Lipschitz nonlinearity was established in [8]. In this paper, we will further extend the method for