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Self-adjoint extensions and stochastic completeness of the Laplace–Beltrami operator on conic and anticonic surfaces [☆]

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Abstract

We study the evolution of the heat and of a free quantum particle (described by the Schrödinger equation) on two-dimensional manifolds endowed with the degenerate Riemannian metric $ds^2 = dx^2 + |x|^{-2\alpha} d\theta^2$, where $x \in \mathbf{R}$, $\theta \in \mathbb{T}$ and the parameter $\alpha \in \mathbf{R}$. For $\alpha \le -1$ this metric describes cone-like manifolds (for $\alpha = -1$ it is a flat cone). For $\alpha = 0$ it is a cylinder. For $\alpha \ge 1$ it is a Grushin-like metric. We show that the Laplace–Beltrami operator Δ is essentially self-adjoint if and only if $\alpha \notin (-3, 1)$. In this case the only self-adjoint extension is the Friedrichs extension Δ_F, that does not allow communication through the singular set $\{x = 0\}$ both for the heat and for a quantum particle. For $\alpha \in (-3, -1]$ we show that for the Schrödinger equation only the average on θ of the wave function can cross the singular set, while the solutions of the only Markovian extension of the heat equation (which indeed is Δ_F) cannot. For $\alpha \in (-1, 1)$ we prove that there exists a canonical self-adjoint extension Δ_B, called bridging extension, which is Markovian and allows the complete communication through the singularity (both of the heat and of a quantum particle). Also, we study the stochastic completeness (i.e., conservation of the L^1 norm for the

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heat equation) of the Markovian extensions Δ_F and Δ_B, proving that Δ_F is stochastically complete at the singularity if and only if $\alpha \leq -1$, while Δ_B is always stochastically complete at the singularity.

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Keywords: Heat and Schrödinger equation; Degenerate Riemannian manifold; Grushin plane; Stochastic completeness

1. Introduction

In this paper we consider the Riemannian metric on $M = \big(\mathbf{R} \setminus \{0\}\big) \times \mathbb{T}$ whose orthonormal basis has the form:

$$X_1(x,\theta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad X_2(x,\theta) = \begin{pmatrix} 0 \\ |x|^\alpha \end{pmatrix}, \quad \alpha \in \mathbf{R}. \tag{1}$$

Here $x \in \mathbf{R} \setminus \{0\}$, $\theta \in \mathbb{T}$ and $\alpha \in \mathbf{R}$ is a parameter. In other words we are interested in the Riemannian manifold (M, g), where

$$g = dx^2 + |x|^{-2\alpha} d\theta^2, \text{ i.e., in matrix notation } g = \begin{pmatrix} 1 & 0 \\ 0 & |x|^{-2\alpha} \end{pmatrix}. \tag{2}$$

Define

$$M_{\text{cylinder}} = \mathbf{R} \times \mathbb{T}, \qquad M_{\text{cone}} = M_{\text{cylinder}} / \sim,$$

where $(x_1, \theta_1) \sim (x_2, \theta_2)$ if and only if $x_1 = x_2 = 0$. In the following we are going to suitably extend the metric structure to M_{cylinder} through (1) when $\alpha \geq 0$, and to M_{cone} through (2) when $\alpha < 0$.

Recall that, on a general two dimensional Riemannian manifold for which there exists a global orthonormal frame, the distance between two points can be defined equivalently as

$$d(q_1, q_2) = \inf \Bigg\{ \int_0^1 \sqrt{u_1(t)^2 + u_2(t)^2}\, dt \mid \gamma : [0,1] \to M \text{ Lipschitz }, \gamma(0) = q_1,\ \gamma(1) = q_2 \\ \text{and } u_1,\ u_2 \text{ are defined by } \dot\gamma(t) = u_1(t) X_1(\gamma(t)) + u_2(t) X_2(\gamma(t)) \Bigg\}, \tag{3}$$

$$d(q_1, q_2) = \inf \left\{ \int_0^1 \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt \mid \gamma : [0,1] \to M \text{ Lipschitz }, \gamma(0) = q_1,\ \gamma(1) = q_2 \right\}, \tag{4}$$

where $\{X_1, X_2\}$ is the global orthonormal frame for (M, g).