Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 3327–3349
Journal of Differential Equations
www.elsevier.com/locate/jde

A singular semilinear problem with dependence on the gradient

Francesca Faraci *, Daniele Puglisi

Department of Mathematics and Computer Sciences, University of Catania, Italy

Received 11 March 2015; revised 29 June 2015

Available online 2 November 2015

Abstract

In the present paper we study the problem of finding classical and weak solutions for a semilinear elliptic equation involving a singular term and a nonlinearity depending on the gradient. The approach we follow, based on a suitable combination of sub–supersolutions techniques and fixed point theory, is not quite the same in the two cases and requires different kind of assumptions.

MSC: 35J91; 35J20; 35D30; 35D35

Keywords: Weak and classical solutions; Singularity; Dependence on the gradient; Variational methods; Fixed point theory

1. Introduction

In the present paper we deal with the existence of classical and weak solutions of a semilinear problem depending on a singular term and on the gradient. Namely, we will study the problem

$$\begin{cases} -\Delta u = g(x,u) + h(x,u,\nabla u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

* Corresponding author.
E-mail addresses: ffaraci@dmi.unict.it (F. Faraci), dpuglisi@dmi.unict.it (D. Puglisi).

http://dx.doi.org/10.1016/j.jde.2015.10.031
0022-0396/

where $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is a bounded domain with smooth boundary $\partial\Omega$, g and h are non-negative functions. We will require that h satisfies a suitable growth assumption and g is positive in a neighborhood of zero. Our assumptions on g cover the case when g exhibits a singularity at zero, i.e.

$$\lim_{s \to 0^+} g(x, s) = +\infty$$

uniformly on a subset of $\overline{\Omega}$ and for this reason we refer to g as the singular term. The model we have in mind is $g(x, s) = a(x)s^{-\gamma}$ with arbitrary $\gamma > 0$ although more general nonlinear terms can be considered.

Definition 1.1. We say that u is a *classical solution* of (P) if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and (P) is satisfied pointwisely.

We say that u is a *weak solution* of (P) if $u \in W_0^{1,2}(\Omega)$, $u > 0$ almost everywhere in Ω, and for all $v \in W_0^{1,2}(\Omega)$ the following conditions hold

$$(g(\cdot, u) + h(\cdot, u, \nabla u))v \in L^1(\Omega),$$
$$\int_\Omega \nabla u \nabla v \, dx = \int_\Omega (g(x, u) + h(x, u, \nabla u))v \, dx.$$

Because of the presence of the singularity a classical solution not necessarily defines a weak solution.

In [8] the existence of a unique classical solution for (P) is obtained when g is a decreasing function (with respect to the real variable) and $h = 0$, via approximation techniques and sub–supersolution methods (see also [13] for an alternative approach). This result has been generalized in [7] in the presence of a sublinear nonlinearity h not depending on the gradient. More precisely, in [7], the authors consider the variational ε-problem

$$\begin{cases} -\Delta u = g(x, u + \varepsilon) + h(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

and select a solution u_ε greater than a subsolution $\underline{u}$ not depending on ε. The limit of this sequence, as $\varepsilon \to 0$, gives a solution of the original problem greater than or equal to $\underline{u}$.

The existence of weak solutions for the singular semilinear problem

$$\begin{cases} -\Delta u = a(x)u^{-\gamma} + \lambda h(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has been investigated in the last years. There is a wide literature in the case $0 < \gamma < 1$ (see [9] and the references therein). In such a case it is possible to define the energy functional, which, although not Gâteaux differentiable, is continuous in $W_0^{1,2}(\Omega)$. This allows to use well known variational techniques such as critical point theory in convex closed sets, non-smooth analysis, sub–supersolutions techniques.