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Estimates for the deviation of solutions and eigenfunctions of second-order elliptic Dirichlet boundary value problems under domain perturbation

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Abstract

Estimates in suitable Lebesgue or Sobolev norms for the deviation of solutions and eigenfunctions of second-order uniformly elliptic Dirichlet boundary value problems subject to domain perturbation in terms of natural distances between the domains are given. The main estimates are formulated via certain natural and easily computable "atlas" distances for domains with Lipschitz continuous boundaries. As a corollary, similar estimates in terms of more "classical" distances such as the Hausdorff distance or the Lebesgue measure of the symmetric difference of domains are derived. Sharper estimates are also proved to hold in smoother classes of domains.

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1. Introduction

In this paper we prove stability estimates for solutions and eigenfunctions of second order uniformly elliptic Dirichlet boundary value problems subject to domain perturbation: we give explicit estimates in L^p-norm and $W^{1,p}$-norm, where p takes values in a suitable subinterval of $[1, \infty]$, which contains 2 as an interior point, of the difference of solutions and eigenfunc-

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tions on different domains of the Euclidean n-dimensional space in terms of suitable distances between the domains such as, e.g., the Hausdorff distance, the Lebesgue measure of the symmetric difference of domains, or even certain atlas distances between domains introduced in the sequel.

In order to describe more precisely our results, let us introduce some notation. Consider a positive symmetric uniformly elliptic linear second-order differential operator in divergence form

$$Su = -\mathrm{div}(A(x)\nabla u) + b(x)u \tag{1.1}$$

in $\mathbb{R}^n$, $n \in \mathbb{N}$, with locally $C^{1,\alpha}$ $(0 < \alpha \le 1)$ coefficients. That is, we are assuming the matrix $A(x)$ entries and $b(x)$ are locally $C^{1,\alpha}$ functions of $x \in \mathbb{R}^n$ for some $0 < \alpha \le 1$, such that $A(x)$ is Hermitian and, for a suitable $\lambda > 0$,

$$A(x) \ge \lambda I_n$$

holds (in the sense of the order relation on Hermitian matrices, I_n is the n-dimensional unit matrix) for all $x \in \mathbb{R}^n$. In addition, we assume $b(x) \ge 0$ for all $x \in \mathbb{R}^n$.

The estimates regarding solutions that we obtain are of the following type: we find or exhibit examples of

- $\mathcal{F}$, a family of domains (bounded nonempty open sets) in $\mathbb{R}^n$,
- $d(\cdot, \cdot)$ a distance on $\mathcal{F}$,
- D, a universal fixed domain that contains all elements of $\mathcal{F}$,
- $\mathcal{G}$ a subfamily of $\mathcal{F}$, to be interpreted as the collection of domains which are being perturbed,
- $\mathcal{G}' = \{\mathcal{G}'_\Omega\}_{\Omega\in\mathcal{G}}$, where, for each $\Omega \in \mathcal{G}$, $\mathcal{G}'_\Omega$ is a subfamily of $\mathcal{F}$, consisting of the so-called *admissible* perturbations of Ω,[1]
- $\mathcal{X}(D)$, $\mathcal{W}(D)$ normed spaces of (possibly, generalized) functions defined on D, such that for each $f \in \mathcal{W}(D)$, there exists a unique solution (in some sense[2]) $u_\Omega \in \mathcal{X}(D)$ to problem

$$\begin{cases} Su = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.2}$$

- a parameter $0 < \gamma$ (≤ 1),

such that theorems of the following kind may be formulated.

[1] At a first reading one may take $\mathcal{G} = \mathcal{F}$ and $\mathcal{G}_\Omega = \mathcal{F}$ for all $\Omega \in \mathcal{G} = \mathcal{F}$. However, such an assumption would imply that our stability result (Theorem 1.1) would be symmetric in Ω and its perturbation Ω'. But this is not always the case: there are results in which (i) we are forced to restrict to a subclass $\mathcal{G}$ of $\mathcal{F}$ of domains whose perturbations we may investigate and (ii) for any $\Omega \in \mathcal{G}$, the class of admissible perturbations of Ω is not the whole $\mathcal{F}$ but rather a subclass $\mathcal{G}_\Omega$, depending on Ω itself.

[2] Usually, it is required that u satisfy equation $Su = f$ in Ω in the sense of distributions, while the boundary values be attained in the sense of traces of Sobolev spaces theory. To be more precise, the problem is uniquely solved by u_Ω in some normed space $\mathcal{X}(\Omega)$ (depending on Ω), and after extending u_Ω to all of D (in this paper by setting $u_\Omega = 0$ on $D \setminus \Omega$), then $u_\Omega \in \mathcal{X}(D)$.