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A generalized regularity criterion for 3D Navier–Stokes equations in terms of one velocity component

Chenyin Qian

Department of Mathematics, Zhejiang Normal University, Jinhua, 321004, China

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Abstract

In this article, we study the global regularity of the 3D Navier–Stokes equations in terms of one velocity component. In particular, we establish a new version of regularity criterion in the framework of the anisotropic Lebesgue space. It is a generalization of the notable work of C. Cao and E.S. Titi (2011) [3].

Keywords: 3D Navier–Stokes equations; Leray–Hopf weak solution; Regularity criterion

1. Introduction

In the present paper, we address sufficient conditions for the regularity of the Leray–Hopf weak solution of the Navier–Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = 0, & \text{in } \mathbb{R}^3, t \geq 0, \\ \nabla\cdot u = 0, & \text{in } \mathbb{R}^3, t \geq 0, \\ u(x,0) = u_0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $u = (u_1, u_2, u_3) : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$ is the velocity field, $p : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ is a scalar pressure and u_0 is the initial velocity field, $\nu > 0$ is the viscosity. We set $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ as the

E-mail addresses: cyqian@zjnu.cn, qcyjcsx@163.com.

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horizontal gradient operator and $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ as the horizontal Laplacian, and Δ and ∇ are the usual Laplacian and the gradient operators, respectively. Here we use the classical notation

$$(u \cdot \nabla)v = \sum_{i=1}^{3} u_i \partial_{x_i} v_k, \ (k = 1, 2, 3), \quad \nabla \cdot u = \sum_{i=1}^{3} \partial_{x_i} u_i,$$

and for the sake of simplicity, we denote ∂_{x_i} by ∂_i.

We set

$$\mathcal{V} = \{\phi : \ \text{the 3D vector valued } C_0^\infty \text{ functions and } \nabla \cdot \phi = 0\},$$

which will form the space of test functions. Let H and V be the closure spaces of $\mathcal{V}$ in L^2 under L^2-topology, and in H^1 under H^1-topology, respectively. We denote by L^q and H^m the usual L^q-Lebesgue and Sobolev spaces, respectively, and by

$$\|\varphi\|_{L^q} = \left(\int_{\mathbb{R}^3} |\varphi(x)|^q \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \right)^{\frac{1}{q}}, \ \text{for every } \varphi \in L^q(\mathbb{R}^3),$$

we also denote by $L^q(\mathbb{R}^2_{x_2,x_3}, L^p(\mathbb{R}_{x_1}))$ the anisotropic Lebesgue space, and by

$$\Big\| \|\varphi\|_{L^p_{x_1}} \Big\|_{L^q_{x_2,x_3}} = \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} |\varphi(x)|^p \mathrm{d}x_1 \right)^{\frac{q}{p}} \mathrm{d}x_2 \mathrm{d}x_3 \right)^{\frac{1}{q}}, \ \text{for every } \varphi \in L^q(\mathbb{R}^2_{x_2,x_3}, L^p(\mathbb{R}_{x_1})).$$

We first recall some of the classical definitions and results on weak solutions of the Navier–Stokes equations.

Definition 1.1. A weak solution of (1.1) on $[0, T]$ (or $[0, \infty)$ if $T = \infty$) is a function $u : [0, T] \to L^2(\mathbb{R}^3)$ in the class

$$u \in C_w([0, T]; L^2(\mathbb{R}^3)) \cap L^2_{loc}(0, T; H^1(\mathbb{R}^3)),$$

satisfying

$$(u(t), \varphi(t)) + \int_0^t \{-(u, \partial_t \varphi) + \nu(\nabla u, \nabla \varphi) + (u \cdot \nabla u, \varphi)\} ds = (u_0, \varphi(0)), \tag{1.2}$$

for all $t \in [0, T]$ and all test functions $\varphi \in C_0^\infty([0, T] \times \mathbb{R}^3)$ with $\nabla \cdot \varphi = 0$. Here $(\cdot, \cdot)$ stands for L^2-inner product, and C_w signifies continuity in the weak topology.