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Liouville theorems, universal estimates and periodic solutions for cooperative parabolic Lotka–Volterra systems

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Abstract

We consider positive solutions of cooperative parabolic Lotka–Volterra systems with equal diffusion coefficients, in bounded and unbounded domains. The systems are complemented by the Dirichlet or Neumann boundary conditions. Under suitable assumptions on the coefficients of the reaction terms, these problems possess both global solutions and solutions which blow up in finite time. We show that any solution (u, v) defined on the time interval $(0, T)$ satisfies a universal estimate of the form

$$u(x,t) + v(x,t) \le C(1 + t^{-1} + (T - t)^{-1}),$$

where C does not depend on x, t, u, v, T. In particular, this bound guarantees global existence and boundedness for threshold solutions lying on the borderline between blow-up and global existence. Moreover, this bound yields optimal blow-up rate estimates for solutions which blow up in finite time. Our estimates are based on new Liouville-type theorems for the corresponding scaling invariant parabolic system and require an optimal restriction on the space dimension n: $n \le 5$. As an application we also prove the existence of time-periodic positive solutions if the coefficients are time-periodic. Our approach can also be used for more general parabolic systems.

Keywords: Lotka–Volterra systems; Parabolic; A priori estimates; Blow-up; Liouville-type theorems; Periodic solutions

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1. Introduction

We consider nonnegative solutions of the Lotka–Volterra system

$$\left.\begin{aligned} u_t - d_1\Delta u &= u(a_1 - b_1 u + c_1 v), \\ v_t - d_2\Delta v &= v(a_2 - b_2 v + c_2 u), \end{aligned}\right\} \qquad x\in\Omega,\ t\in(0,T), \tag{1}$$

where Ω is a (possibly unbounded) domain in $\mathbb{R}^n$ with a uniformly C^2 smooth boundary $\partial\Omega$, $T \in (0,\infty]$, d_1, d_2 are positive constants and

$$a_i, b_i, c_i \in L^\infty(\Omega\times(0,\infty)) \ \text{ for } \ i=1,2. \tag{2}$$

Except for some marginal results in Theorem 7 and Remark 8(i) we will always assume

$$b_1, b_2, c_1, c_2 > 0, \qquad c_1c_2 > b_1b_2. \tag{3}$$

Let us mention that if $c_1, c_2 > 0$ then the case $c_1c_2 > b_1b_2$ (or $c_1c_2 < b_1b_2$) is called the strong (or weak) mutualistic case, respectively, see [12,7]. If the coefficients a_i, b_i, c_i are constant and system (1) is complemented by the homogeneous Dirichlet or Neumann boundary conditions then all solutions are global and bounded in the weak mutualistic case while some solutions blow up in finite time in the strong mutualistic case (see [14, Section 12.6]).

By ν we denote the outer unit normal on $\partial\Omega$ and by $(1)_D$ or $(1)_N$ we denote system (1) complemented by the Dirichlet boundary conditions $u=v=0$ on $\partial\Omega\times(0,T)$ or the Neumann boundary conditions $u_\nu = v_\nu = 0$ on $\partial\Omega\times(0,T)$, respectively. Notice that $(1)_D = (1)_N = (1)$ if $\Omega = \mathbb{R}^n$. If Ω is bounded then by Λ_1 we denote the least eigenvalue of the negative Dirichlet Laplacian in Ω. Except for Section 5, by a solution we will always mean a nonnegative classical solution.

First consider the Dirichlet problem $(1)_D$ and assume that Ω is bounded and the coefficients a_i, b_i, c_i are constant. Then some solutions of $(1)_D$ blow up in finite time and the blow-up rates of such solutions have been studied in [9,8]. However, an upper estimate of the blow-up rate (which is usually much more difficult than a lower estimate) has only been established if $n=1$. Under suitable additional assumptions, $(1)_D$ possesses also nontrivial global solutions and steady states: If we assume $a_1/d_1, a_2/d_2 < \Lambda_1$, for example, then the existence of global solutions follows from the stability of the zero solution. If, in addition, $n \le 5$ then there exists a positive steady state, and the assumption $n\le 5$ is also necessary if Ω is starshaped and $a_1/d_1 = a_2/d_2$, see [11].

If one considers the Neumann problem $(1)_N$ with Ω bounded and a_i, b_i, c_i constant then some solutions blow up again and even "diffusion-induced blow-up" occurs: There exist blow-up solutions of $(1)_N$ such that the solutions of the corresponding system of ODEs exist globally, see [12]. On the other hand, nontrivial global solutions also exist if $a_1, a_2 < 0$, for example.

The existence of blow-up and global solutions of system (1) is also known in the case of non-constant coefficients, see [10] and the references therein, for example. If the problem $(1)_D$ or $(1)_N$ possesses both global solutions and solutions which blow up in finite time then one can study so-called threshold solutions, i.e. solutions lying on the borderline between global existence and blow-up. The study of such solutions is difficult even for the scalar problem

$$\left.\begin{aligned} u_t - \Delta u &= cu^2, \qquad && x\in\Omega,\ t>0, \\ u &= 0, && x\in\partial\Omega,\ t>0, \end{aligned}\right\} \tag{4}$$