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Global well-posedness and grow-up rate of solutions for a sublinear pseudoparabolic equation

Sujin Khomrutai

Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand

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Abstract

We study positive solutions of the pseudoparabolic equation with a sublinear source in $\mathbb{R}^n$. In this work, the source coefficient (or potential) can be unbounded and time-dependent. Global existence of solutions to the Cauchy problem is established within weighted continuous spaces by approximation and a monotonicity argument. Every solution with a non-zero initial value is shown to exhibit a certain lower grow-up and radial growth bound, depending only upon the sublinearity and the potential. We prove the key comparison principle, using the lower grow-up and growth bound, and then settle the uniqueness of solutions for the problem with a non-zero initial value. For the zero initial-valued problem, we classify all non-trivial solutions in terms of the maximal solution. Finally, when the initial value has a power radial growth at infinity, we can derive the precise grow-up rate of solutions and obtain the critical growth exponent.

MSC: 35A01; 35A02; 35A09; 35B09; 35B33; 35B40; 35B51; 35K70

Keywords: Sublinear pseudoparabolic equation; Non-autonomous and unbounded source; Comparison principle; Global well-posedness; Grow-up rate; Critical exponent

1. Introduction

We study the existence, uniqueness or non-uniqueness, and grow-up rate of solutions $u = u(x,t) \geq 0$ to the semilinear pseudoparabolic Cauchy problem

E-mail address: sujin.k@chula.ac.th.

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$$\begin{cases} \partial_t u - \triangle \partial_t u = \triangle u + V(x,t)u^p & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $0 < p < 1$ is a constant, $V, u_0 \geq 0$ are given continuous functions, and $n \geq 1$. In this work, the potential function V can be non-autonomous and unbounded. Our framework can be applied to a more general equation of the form

$$\partial_t u - \nu \triangle \partial_t u = \triangle u + S(x,t,u) \tag{1.2}$$

where $\nu \geq 0$ is a constant and the source function S satisfies $0 \leq S(x,t,u) \lesssim V(x,t)|u|^p$. This equation, of course, includes the sublinear heat equations with non-autonomous unbounded source.

Pseudoparabolic equations are models of many important nonlinear physical systems (see for instance [2,5,6,18,20] and the references therein). Also, in recent years, there has been a great interest in studying nonlinear evolution equations with unbounded, or singular, and even time-dependent coefficients.

When the viscosity term $\triangle \partial_t u$ is dropped, Eq. (1.1) becomes the heat equation which is closely related with the pseudoparabolic equation [21–24]. Nonlinear heat equations and systems on a (bounded or unbounded) domain have been studied extensively and are quite well understood. However, the nonlinear pseudoparabolic equations, especially those considered on an unbounded domain, have received very little attention. This could be explained from the difference of their Green kernels:

$$H(x,t) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2}\right), \qquad G(x,t) = \mathcal{F}^{-1}\left(e^{-t|\xi|^2/(1+|\xi|^2)}\right) \tag{1.3}$$

for the heat equation and the pseudoparabolic equation, respectively. We note that if $u \in \mathcal{S}'$, a tempered distribution, then $H * u \in \mathcal{S}$, however, we only have $G * u \in \mathcal{S}'$ [13]. In addition, $H \simeq t^{-n/2} e^{-|x|^2/(4t)}$ but there is no explicit expression for the pseudoparabolic kernel. In fact, the kernel G is very complicated, see Eq. (2.5).

To our knowledge, the first study on positive solutions of semilinear pseudoparabolic equations on $\mathbb{R}^n$ is the work by Cao et al. [8] (see also [14] for an inspiring work on real value solutions). They considered Eq. (1.1) with a constant potential, that is

$$\partial_t u - \triangle \partial_t u = \triangle u + u^p \qquad \text{in } \mathbb{R}^n \times (0,\infty), \tag{1.4}$$

for any constant $p > 0$. In the sublinear case, that is $0 < p < 1$, they established the existence of solutions within $BC(\mathbb{R}^n) \triangleq C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ by the method of sub- and super-solutions, but left the uniqueness of solutions an open problem. Recently, this problem was settled in [16]. It was found that the uniqueness is guaranteed provided that the initial condition is non-zero, whereas all non-trivial solutions with the zero initial value are shown to be the delays of the maximal solution $u_* = ((1-p)t)^{1/(1-p)}$. It is remarkable that these are the same findings as that for the sublinear heat equation [1].

Let us explain our main results. In this work, we extend the results of [8] and [16]. The solutions to Eq. (1.1) are studied in the non-local (or mild) formulation

$$u(x,t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-\tau)\mathcal{B}\left(Vu^p\right)(x,\tau)\,d\tau, \tag{1.5}$$