Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 3749–3784
Journal of Differential Equations
www.elsevier.com/locate/jde

A new index theory for linear self-adjoint operator equations and its applications

Qi Wang [a,1], Chungen Liu [b,*,2]

[a] *Institute of Contemporary Mathematics, School of Mathematics and Information Science, Henan University, Kaifeng 475000, PR China*

[b] *School of Mathematics and LPMC, Nankai University, Tianjin 300071, PR China*

Received 19 April 2015; revised 23 October 2015

Available online 12 November 2015

Abstract

We develop an index theory for linear self-adjoint operator equation with the linear self-adjoint operator having no compact resolvent. As applications, we consider the existence and multiplicity of periodic solutions of wave equations and beam equations.

MSC: 58E05; 34B15; 35B10; 35L05; 70H05

Keywords: Index theory; Dual variational methods; Wave equation

1. Introduction

Many problems can be displayed as a self-adjoint operator equation

$$Au = F'(u),\ u \in D(A) \subset H, \tag{O.E.}$$

* Corresponding author.
E-mail addresses: mathwq@henu.edu.cn (Q. Wang), liucg@nankai.edu.cn (C. Liu).
[1] Partially supported by NNSF of China (11301148).
[2] Partially supported by NNSF of China (10071140, 11471170) and 973 program of STM (2011CB808002) and SRFDP.

http://dx.doi.org/10.1016/j.jde.2015.10.046
0022-0396/

where H is an infinite-dimensional separable Hilbert space, A is a self-adjoint operator on H with its domain $D(A)$, F is a nonlinear functional on H, such as Dirichlet problem for Laplace's equation on bounded domain, periodic solutions of Hamiltonian systems, Schrödinger equation, periodic solutions of wave equation and so on. By variational method, we know that the solutions of (O.E.) correspond to the critical points of a functional on a Hilbert space. So we can transform the problem of finding the solutions of (O.E.) into the problem of finding the critical points of a functional. Many theories have been developed to do so. Among these theories, Morse theory is one of the remarkable theories, and it has a great advantage in displaying the relationship between the global and local behavior of the functional. Morse theory can be used directly in Dirichlet problem for Laplace's equation on bounded domain and periodic solutions of second order Hamiltonian systems, since the Morse indices of the critical points are finite. But for the problems of periodic solutions of first order Hamiltonian systems, Schrödinger equations, wave equations, Morse theory cannot be used directly because in these situations the functionals are strongly indefinite in the sense that they are unbounded from above and below and the Morse indices at the critical points of these functionals are infinite. Fortunately, some methods have been developed to deal with these situations, such as Galerkin approximation methods, saddle point reduction (a kind of Lyapunov–Schmidt procedure, see e.g. Amann [2], Amann and Zehnder [4] and Chang [9]), dual variational methods and convex analysis theory (see e.g. Aubin and Ekeland [5], Ekeland [14], Ekeland and Temam [17]). By these methods, the solutions of (O.E.) correspond to the critical points of functionals with finite relative Morse indices, then we can use Morse theory to find the solutions of (O.E.).

Related to Morse theory, the index theory is worth to note here. By the work [13] of I. Ekeland, an index theory for convex linear Hamiltonian systems was established. By the works [7,28–30] of Conley, Zehnder and Long, an index theory for symplectic paths was introduced. These index theories have important and extensive applications, e.g. [11,15,16,25,33]. In [31,32] Long and Zhu defined spectral flows for paths of linear operators and redefined Maslov index for symplectic paths. Additionally, Abbondandolo defined a relative Morse index for Fredholm operator with compact perturbation (see [1] and the references therein). Chen and Hu defined the Maslov index for homoclinic orbits in [6]. In the study of the L-solutions (the solutions starting and ending at the same Lagrangian subspace L) of Hamiltonian systems, the second author of this paper introduced in [23] an index theory for symplectic paths using the algebraic methods and gave some applications in [23,24]. Then this index had been generalized by the authors of this paper and Lin in [26].

In addition to the above index theories defined for specific forms, Dong in [12] developed an index theory for abstract operator equations (O.E.). As an essential condition, he assumed that the embedding $D(A) \hookrightarrow H$ was compact. As applications, he considered the second order Hamiltonian systems, elliptic partial differential equations and first order Hamiltonian systems. Recently, the authors of this paper in [37,38] defined their index theory for abstract operator equations (O.E.) by relative Fredholm index and spectrum flow. We also needed the condition of compact embedding $D(A) \hookrightarrow H$. As applications, we considered delay differential system and a kind of infinite dimensional Hamiltonian systems. But for the cases of wave equations, beam equations and so on, the above two index theories will not work, since the corresponding operators have essential spectrum and the condition of compact embedding will not be satisfied.

In order to overcome this difficulty, in this paper, we will develop a new index theory for (O.E.) where the essential spectrum of operator A may not be empty and show that this index theory is a natural generalization of the above two index theories, i.e., these index theories will coincide with each other up to a constant when the operator A has no essential spectrum.