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J. Differential Equations 260 (2016) 3785–3843

Journal of Differential Equations

www.elsevier.com/locate/jde

Analysis of a slow–fast system near a cusp singularity

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Received 29 June 2015; revised 22 October 2015

Available online 17 November 2015

Abstract

This paper studies a slow–fast system whose principal characteristic is that the slow manifold is given by the critical set of the cusp catastrophe. Our analysis consists of two main parts: first, we recall a formal normal form suitable for systems as the one studied here; afterwards, taking advantage of this normal form, we investigate the transition near the cusp singularity by means of the blow up technique. Our contribution relies heavily in the usage of normal form theory, allowing us to refine previous results.

Contents

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http://dx.doi.org/10.1016/j.jde.2015.10.045

1. Introduction

A *slow–fast system* (SFS) is a singularly perturbed ordinary differential equation of the form

$$\begin{aligned} \dot{x} &= f(x, z, \varepsilon) \\ \varepsilon \dot{z} &= g(x, z, \varepsilon), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ are local coordinates and where $\varepsilon > 0$ is a small parameter. The over-dot denotes the derivative with respect to the time parameter t. Throughout this text, we assume that the functions f and g are of class $\mathcal{C}^\infty$. In applications (e.g. [25]), $z(t)$ represents states or measurable quantities of a process while $x(t)$ stands for control parameters. The parameter ε models the difference of the rates of change between the variables z and x. That is why systems like (1) are often used to model phenomena with two time scales. Observe that the smaller ε is, the faster z evolves with respect to x. Therefore we refer to x (resp. z) as the *slow* (resp. *fast*) variable. The time parameter t is known as the *slow time*. For $\varepsilon \neq 0$, we can define a new time parameter τ by the relation $t = \varepsilon\tau$. With this time reparametrization (1) can be written as

$$\begin{aligned} x' &= \varepsilon f(x, z, \varepsilon) \\ z' &= g(x, z, \varepsilon), \end{aligned} \tag{2}$$

where now the prime denotes the derivative with respect to the rescaled time parameter τ, which we call *the fast time*. Since we consider only autonomous systems, we often omit to indicate the time dependence of the variables. In the rest of this document, we prefer to work with slow–fast systems presented as (2).

Observe that as long as $\varepsilon \neq 0$ and f is not identically zero, systems (1) and (2) are equivalent. A first approach to understand the qualitative behavior of slow–fast systems is to study the limit $\varepsilon \to 0$. The slow equation (1) restricted to $\varepsilon = 0$ reads as