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Cubic perturbations of elliptic Hamiltonian vector fields of degree three

Lubomir Gavrilov [a,*], Iliya D. Iliev [b]

[a] Institut de Mathématiques de Toulouse, Université de Toulouse, F-31062 Toulouse, France
[b] Institute of Mathematics, Bulgarian Academy of Sciences, Bl. 8, 1113 Sofia, Bulgaria

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Abstract

The purpose of the present paper is to study the limit cycles of one-parameter perturbed plane Hamiltonian vector field X_ε

$$X_\varepsilon : \begin{cases} \dot{x} = \ \ H_y + \varepsilon f(x,y) \\ \dot{y} = -H_x + \varepsilon g(x,y), \end{cases} \qquad H = \frac{1}{2}y^2 + U(x)$$

which bifurcate from the period annuli of X_0 for sufficiently small ε. Here U is a univariate polynomial of degree four without symmetry, and f, g are *arbitrary* cubic polynomials in two variables.

We take a period annulus and parameterize the related displacement map $d(h,\varepsilon)$ by the Hamiltonian value h and by the small parameter ε. Let $M_k(h)$ be the k-th coefficient in its expansion with respect to ε. We establish the general form of M_k and study its zeroes. We deduce that the period annuli of X_0 can produce for sufficiently small ε, at most 5, 7 or 8 zeroes in the interior eight-loop case, the saddle-loop case, and the exterior eight-loop case respectively. In the interior eight-loop case the bound is exact, while in the saddle-loop case we provide examples of Hamiltonian fields which produce 6 small-amplitude limit cycles. Polynomial perturbations of X_0 of higher degrees are also studied.

* Corresponding author.
E-mail address: lubomir.gavrilov@math.univ-toulouse.fr (L. Gavrilov).

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Contents

1. Introduction

We consider cubic systems in the plane which are small perturbations of Hamiltonian systems with a center. Our goal is to estimate the number of limit cycles produced by the perturbation. The Hamiltonians we consider have the form $H = y^2 + U(x)$ where U is a polynomial of degree 4. In this paper we exclude from consideration the four symmetric Hamiltonians $H = y^2 + x^2 \pm x^4$, $H = y^2 - x^2 + x^4$ and $H = y^2 + x^4$ because they require a special treatment, see [6]. Therefore, one can use the following normal form of the Hamiltonian

$$H = \frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{a}{4}x^4, \quad a \neq 0, \frac{8}{9}. \tag{1}$$

An easy observation shows that the following four topologically different cases occur:

$a < 0$	saddle-loop,
$0 < a < 1$	eight loop,
$a = 1$	cuspidal loop,
$a > 1$	global center.

There is one period annulus in the saddle-loop and the global center cases, two annuli in the cuspidal loop case, and three annuli in the eight loop case. Note that $a = \frac{8}{9}$ is the symmetric eight loop case which we are not going to deal with. Take small $\varepsilon > 0$ and consider the following one-parameter perturbation of the Hamiltonian vector field associated to H:

$$\begin{aligned} \dot{x} &= H_y + \varepsilon f(x, y), \\ \dot{y} &= -H_x + \varepsilon g(x, y), \end{aligned} \tag{2}$$