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On the inverse nodal problems for discontinuous Sturm–Liouville operators

Yu Ping Wang [a,*], V.A. Yurko [b]

[a] Department of Applied Mathematics, Nanjing Forestry University, Nanjing, 210037, Jiangsu, People's Republic of China

[b] Department of Mathematics, Saratov University, Astrakhanskaya 83, Saratov 410012, Russia

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Abstract

In this paper, we discuss the inverse nodal problems for discontinuous Sturm–Liouville operators with Robin boundary conditions. We show that the potential q up to its mean value on the interval $[0, 1]$ and coefficients h, H, $\frac{b}{a}$ can be uniquely determined by the twin-dense nodal subset on the subinterval $[a_0, b_0]$, $a_0 < \frac{1}{2} < b_0$. In addition, we still establish several uniqueness theorems by the left twin-dense nodal subset on the interval $\left[a_0, \frac{1}{2}\right]$ and additional information, respectively.

MSC: primary 34A55; secondary 34L20, 47E05

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1. Introduction

Consider the following discontinuous Sturm–Liouville operator $L := L(q, h, H, a, b)$ defined by

$$lu = -u'' + q(x)u = \lambda u, \quad x \in (0, 1) \tag{1.1}$$

* Corresponding author.
E-mail addresses: ypwang@njfu.com.cn (Y.P. Wang), yurkova@info.sgu.ru (V.A. Yurko).

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with boundary conditions

$$U_0(u) := u'(0, \lambda) - hu(0, \lambda) = 0, \tag{1.2}$$

$$U_1(u) := u'(1, \lambda) + Hu(1, \lambda) = 0, \tag{1.3}$$

and with the jump conditions

$$\begin{cases} V_1(u) := u\left(\frac{1}{2}+, \lambda\right) - au\left(\frac{1}{2}-, \lambda\right) = 0, \\ V_2(u) := u'\left(\frac{1}{2}+, \lambda\right) - a^{-1}u'\left(\frac{1}{2} - 0, \lambda\right) - bu\left(\frac{1}{2}-, \lambda\right) = 0, \end{cases} \tag{1.4}$$

where λ is the spectral parameter, $h, H, a, b \in \mathbb{R}$, $a > 0$, $q(x)$ is a real-valued function and $q \in L^1[0, 1]$.

Hald [12] first discussed the half inverse problem for (1.1)–(1.4), which appears in the torsional modes of the Earth (see [1,12–14]), and showed that if q is prescribed on $\left[0, \frac{1}{2}\right]$ and coefficient h of the boundary condition is given a priori, then only one spectrum is sufficient to determine the potential q on the interval $[0, 1]$ and coefficients H, a, b. Yurko [35] verified that all eigenvalues of the operator L are real and simple and addressed the inverse problems for (1.1)–(1.4) either from Weyl function, or from two spectra, or from one spectrum and the corresponding norming constants. More related results for discontinuous Sturm–Liouville operators were studied by many authors (see [1,12–14,27,30,35]). Inverse spectral problems for the classical Sturm–Liouville operator, i.e., (1.1)–(1.3) without discontinuities, have been studied fairly completely (see [3,4,10,16,21–23,29] and other works). For example, Marchenko [22] showed that the Weyl m-function of the classical Sturm–Liouville operator uniquely determined the coefficients h, H of the boundary conditions as well as the potential q.

The inverse nodal problem is to reconstruct this operator from the given nodal points (zeros) of its eigenfunctions. Inverse nodal problems for various differential operators were investigated by a number of authors (see [2,5–9,11,15,17–20,24–28,30–34,36] and the references therein). For (1.1)–(1.3), McLaughlin [24] showed that the potential q up to its mean value, and coefficients h, H of boundary conditions can be uniquely determined by a dense subset of nodal points of its eigenfunctions. X.F. Yang [33] verified that the potential q up to its mean value, and coefficients h, H of boundary conditions can be uniquely determined by the s-dense nodal set on the interval $[0, b_0]$, $\frac{1}{2} < b_0 \le 1$. Then, Cheng et al. [9] improved the Yang's theorem by using the condition of twin-dense instead of s-dense. Later, Guo and Wei [11] showed that the twin-dense nodal set on the subinterval $[a_0, b_0]$, $a_0 < \frac{1}{2} < b_0$, is sufficient to determine the potential q up to its mean value, and coefficients h, H of boundary conditions, which is an improvement of the Yang's theorem in [33] and $b_0 - a_0$ might be arbitrarily small under certain conditions. C.F. Yang [31] proved that "the question if the result holds true for $b \in (0, \frac{1}{2}]$" proposed by X.F. Yang [33] has a negative answer for $b_0 < \frac{1}{2}$ by a counterexample, which illustrates that two operators have the same spectrum and in the set $[0, \frac{1-\varepsilon}{2}] \bigcup \left[\frac{1+\varepsilon}{2}, 1\right]$ for any ε, $0 < \varepsilon < 1$, their nodal points are the same, but $q(x) - \int_0^1 q(x)dx \neq \tilde{q}(x) - \int_0^1 \tilde{q}(x)dx$ on the interval $\left[\frac{1-\varepsilon}{2}, \frac{1+\varepsilon}{2}\right]$. In 2008, Shieh and Yurko [27] explored the inverse nodal problem for (1.1)–(1.4) from the dense nodal set on the interval $[0, b_0]$. Later, C.F. Yang [30] also studied the uniqueness, reconstruction and stability problems for (1.1)–(1.4) from the twin-dense nodal set on the interval $[0, 1]$. However, we are motivated by inverse nodal problems for (1.1)–(1.4) with the twin-dense nodal subset on