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Averaging methods of arbitrary order, periodic solutions and integrability

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Abstract

In this paper we provide an arbitrary order averaging theory for higher dimensional periodic analytic differential systems. This result extends and improves results on averaging theory of periodic analytic differential systems, and it unifies many different kinds of averaging methods.

Applying our theory to autonomous analytic differential systems, we obtain some conditions on the existence of limit cycles and integrability.

For polynomial differential systems with a singularity at the origin having a pair of pure imaginary eigenvalues, we prove that there always exists a positive number N such that if its first N averaging functions vanish, then all averaging functions vanish, and consequently there exists a neighborhood of the origin filled with periodic orbits. Consequently if all averaging functions vanish, the origin is a center for $n = 2$.

Furthermore, in a punctured neighborhood of the origin, the system is C^∞ completely integrable for $n > 2$ provided that each periodic orbit has a trivial holonomy.

Finally we develop an averaging theory for studying limit cycle bifurcations and the integrability of planar polynomial differential systems near a nilpotent monodromic singularity and some degenerate monodromic singularities.

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1. Introduction and statement of the main results

To know when a differential system has or not periodic solutions is very important for understanding its dynamics. Averaging theory is a good theory for studying the periodic solutions. Of course, the averaging theory is a classical tool for studying the behavior of nonlinear differential systems. This theory has a long history that starts with the works of Lagrange and Laplace, who work with it in an intuitive way. One of first formalizations of the averaging theory was done by Fatou in 1928 [15]. Later on Bogoliubov and Krylov [4] in the 1930s and Bogoliubov [3] in 1945 did very important practical and theoretical contributions to the averaging theory. The ideas of averaging theory have been improved in several directions for the finite and infinite dimensional differentiable systems. More recently Hale also did good contributions to the averaging theory, see the books [20,21]. For modern expositions and results on the averaging theory see also the books of Sanders, Verhulst and Murdock [37] and Verhulst [38].

Consider periodic analytic differential systems

$$\dot{x} = F_0(t,x) + \sum_{i=1}^{\infty} \varepsilon^i F_i(t,x), \qquad (t,x) \in \mathbb{R} \times \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is an open subset, ε is a parameter with $|\varepsilon|$ sufficiently small, and the $F_i(t,x)$'s are n-dimensional vector valued analytic functions in their variables in $\mathbb{R} \times \Omega$ and periodic of period T in the variable t. For $z \in \Omega$, let $x(t,z,\varepsilon)$ be the solution of (1) satisfying $x(0,z,\varepsilon) = z$. One of the important problems in the study of the dynamics of a differential system (1) is to know when $x(t,z,\varepsilon)$ is a periodic solution. There are many different methods for studying this problem, and the averaging method is very useful one.

In order to apply the averaging theory for studying periodic orbits of the differential system (1), one of the basic assumptions is that the unperturbed system $\dot{x} = F_0(t,x)$ has an invariant manifold formed by periodic orbits. In this direction there are extensive studies for the first, second and third order averaging theories, see for instance [6–9,13,31,36,37] and the references therein. Also the averaging theory has broad applications, see e.g. [1,2,13,16,17,19,23,28,29,35] and the references therein.

Recently the averaging theory was extended to arbitrary order for computing periodic orbits. Giné et al. [18] provided an arbitrary order averaging formula of system (1) when $n = 1$. In [16] the averaging theory in $\mathbb{R}^n$ up to any order in ε for the particular case $F_0(t,x) \equiv 0$ is described in a recursive way and it is applied to the center problem for planar systems. Llibre et al. [24] further extended the arbitrary order averaging method to any finite dimensional periodic differential system provided that the manifold formed by periodic solutions of the unperturbed differential system $\dot{x} = F_0(t,x)$ is an open subset of Ω. When the manifold formed by periodic solutions of the unperturbed differential system $\dot{x} = F_0(t,x)$ has dimension less than n, Malkin [31] and Roseau [36] provided the averaging theory of first order. For a different and shorter proof, see [6]. Buică et al. [7,8] extended the Malkin and Roseau's first order averaging theory to second order. Here we extend these previous results to arbitrary order for any finite dimensional periodic