Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 4393–4409
Journal of Differential Equations
www.elsevier.com/locate/jde

On chaotic minimal center of attraction of a Lagrange stable motion for topological semi flows

Xiongping Dai

Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

Received 5 August 2015; revised 13 October 2015

Available online 28 November 2015

Abstract

Let $f: \mathbb{R}_+ \times X \to X$ be a topological semi flow on a Polish space X. In 1977, Karl Sigmund conjectured that if there is a point $\boldsymbol{x}$ in X such that the motion $f(\boldsymbol{\cdot}, \boldsymbol{x})$ has just X as its minimal center of attraction, then all such $\boldsymbol{x}$ form a *residual* subset of X. In this paper, we first present a positive solution to this conjecture and then apply it to the study of chaotic dynamics occurring inside or near the minimal center of attraction of a motion $f(\boldsymbol{\cdot}, x)$.

MSC: 37B05; 37A60; 54H20

Keywords: Minimal center of attraction; Chaotic motion; Quasi-weakly almost periodic motion

1. Introduction

By a C^0-*semi flow* over a metric space X, we here mean a transformation $f: \mathbb{R}_+ \times X \to X$ where $\mathbb{R}_+ = [0, \infty)$, which satisfies the following three conditions:

(1) The initial condition: $f(0, x) = x$ for all $x \in X$.
(2) The condition of continuity: if $t_n \to t$ in $\mathbb{R}_+$ and $x_n \to x$ in X, then $f(t_n, x_n) \to f(t, x)$ as $n \to \infty$.

E-mail address: xpdai@nju.edu.cn.

http://dx.doi.org/10.1016/j.jde.2015.11.019
0022-0396/

(3) The semigroup condition: $f(t_2, f(t_1, x)) = f(t_1 + t_2, x)$ for any x in X and any times t_1, t_2 in $\mathbb{R}_+$.

We sometimes write $f(t, x) = f^t(x)$ for any $t \geq 0$ and $x \in X$; and for any given point $x \in X$ we shall call $f(\bullet, x)$ a *motion* in X and $\mathcal{O}_f(x) = f(\mathbb{R}_+, x)$ the *orbit* starting from x. If $\mathcal{O}_f(x)$ is precompact (i.e. $\overline{\mathcal{O}_f(x)}$ is compact) in X, then we say the motion $f(\bullet, x)$ is *Lagrange stable* (cf. [17]).

We refer to any subset Λ of X as an *f-invariant set* if $f(t, x) \in \Lambda$ for each point $x \in \Lambda$ and any time $t \geq 0$. In dynamical systems, statistical mechanics and ergodic theory, we shall have to do with "probability of sojourn of a motion $f(\bullet, x)$ in a given region E of X" as $t \to +\infty$:

$$\boldsymbol{P}(f(\bullet, x) \in E) = \lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathbb{1}_E(f(t, x))dt,$$

where $\mathbb{1}_E(x)$ is the characteristic function of the set E on X. This motivates H.F. Hilmy to introduce following important concept, which was discussed in [15,17,19,14], for example.

Definition 1.1. (See Hilmy, 1936 [12].) Given any $x \in X$, a closed subset C_x of X is called the *center of attraction of the motion* $f(\bullet, x)$ as $t \to +\infty$ if $\boldsymbol{P}(f(\bullet, x) \in B_\varepsilon(C_x)) = 1$ for all $\varepsilon > 0$. If the set C_x does not admit any proper subset which is likewise a center of attraction of the motion $f(\bullet, x)$ as $t \to +\infty$, then C_x is called the *minimal center of attraction of the motion* $f(\bullet, x)$ as $t \to +\infty$. Here $B_\varepsilon(C_x)$ denotes the ε-neighborhood around C_x in X.

First of all, by the classical Cantor–Baire theorem and Zorn's lemma we can obtain the following basic existence of minimal center of attraction of a motion $f(\bullet, x)$.

Lemma 1.2. *Let $f: \mathbb{R}_+ \times X \to X$ be a C^0-semi flow on a metric space X. Then each Lagrange stable motion $f(\bullet, x)$ always possesses the minimal center of attraction.*

From now on, by $\mathcal{C}_x$ we will understand the minimal center of attraction of a Lagrange stable motion $f(\bullet, x)$ as $t \to +\infty$. In [19], Karl Sigmund gave an intrinsic characterization for $\mathcal{C}_x$. In this paper, we shall study the generic property and chaotic dynamics occurring in and near $\mathcal{C}_x$ for a Lagrange stable motion $f(\bullet, x)$.

Just as the existence of one point which is topologically transitive implies that a residual set consists of topologically transitive points, in 1977 Karl Sigmund raised the following open problem:

Conjecture 1.3. *(See K. Sigmund, 1977 [19, Remark 4].) For any homeomorphism f of a compact metric space X, the set of points $x \in X$ with $\mathcal{C}_x = X$, if nonempty, is residual in X.*

Since a residual set contains a dense G_δ subset of X, it is very large from the viewpoint of topology. Although Sigmund's conjecture is of interest, there has not been any progresses on it since 1977 except f satisfies the "specification" property [19, Proposition 6]. In Section 2, we will present a positive solution to Sigmund's conjecture without any imposed shadowing assumption like specification, which may be stated as follows: