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Decay characterization of solutions to Navier–Stokes–Voigt equations in terms of the initial datum

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Abstract

The Navier–Stokes–Voigt equations are a regularization of the Navier–Stokes equations that share some of its asymptotic and statistical properties and have been used in direct numerical simulations of the latter. In this article we characterize the decay rate of solutions to the Navier–Stokes–Voigt equations in terms of the decay character of the initial datum and study the long time behavior of its solutions by comparing them to solutions to the linear part.

MSC: 35B40; 35Q35

Keywords: Navier–Stokes–Voigt equations; Temporal decay estimates; Fourier Splitting method

1. Introduction

The Navier–Stokes–Voigt equations

$$\begin{aligned} \partial_t \left(u - \alpha^2 \Delta u \right) - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, \\ \nabla \cdot u = 0, \\ u_0(x) = u(x,0), \end{aligned} \tag{1.1}$$

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with $\nu > 0$, describe the time evolution of the velocity u and the pressure p of a homogeneous incompressible viscoelastic viscous fluid. These equations were originally used by Oskolkov [19] to study the motion of flows with relaxation time, i.e. the time that the flow takes to respond to an external force, of order α^2/ν and have been proposed by Cao, Lunasin and Titi [5] as a good model for numerical simulations of the Navier–Stokes equations (i.e. (1.1) with $\alpha = 0$), provided $\alpha > 0$ is small.

The presence of the regularizing term $-\alpha^2 \partial_t \Delta u$ in (1.1) has many important consequences. We first note that, assuming u is a regular enough solution, (1.1) has a natural energy equality in the H^1 norm

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|^2_{L^2(\mathbb{R}^3)} + \alpha^2 \|\nabla u(t)\|^2_{L^2(\mathbb{R}^3)}\right) = -\nu \|\nabla u(t)\|^2_{L^2(\mathbb{R}^3)}, \tag{1.2}$$

instead of the usual one for the L^2 norm of the Navier–Stokes equations. Moreover, Navier–Stokes–Voigt equations do not have a parabolic character, as Navier–Stokes equations do, behaving instead as a damped hyperbolic system. This leads to wellposedness for equation (1.1) both forward and backwards in time (see Cao, Lunasin and Titi [5], Kalantarov and Titi [10]) and to differences in the energy spectrum when compared to Navier–Stokes equations (see Levant, Ramos and Titi [16], Ramos and Titi [20]). However, solutions to Navier–Stokes–Voigt equations share many other statistical and asymptotic properties with the Navier–Stokes equations when $\alpha > 0$ is small (see Kalantarov, Levant and Titi [9], Larios and Titi [13], Levant, Ramos and Titi [16], Ramos and Titi [20]), while presenting advantages for the implementation of numerical models for their study. This supports the idea of using (1.1) in direct numerical simulations (see Borges and Ramos [3], Kuberry, Larios, Rebholz and Wilson [12], Layton and Rebholz [14]). For more results concerning the Navier–Stokes–Voigt equations, see Bersellli and Bisconti [1], Ebrahimi, Holst and Lunasin [6], García-Luengo, Marín-Rubio and Real [8], Gao and Sun [7], Tang [24], Yue and Zhong [26] and the references therein.

Only recently, Zhao and Zhu [27] addressed questions concerning the Navier–Stokes–Voigt equations in the whole space $\mathbb{R}^3$. In their work, they proved existence of weak solutions to (1.1), but their main result concerns the decay rate for the H^1 norm of solutions, which, for initial data u_0 in $L^1(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$, is proved to be

$$\|u(t)\|^2_{H^1(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}}. \tag{1.3}$$

The main tool in the proof of these estimates is a clever adaptation of the Fourier Splitting method of M.E. Schonbek [21–23], which takes into account the extra linear term present in (1.1).

The main goal of this article is to prove a decay estimate for the Navier–Stokes–Voigt equations for *any* initial data in $H^1_\alpha(\mathbb{R}^3)$, where

$$\|u\|^2_{H^1_\alpha(\mathbb{R}^3)} = \|u\|^2_{L^2(\mathbb{R}^3)} + \alpha^2 \|\nabla u\|^2_{L^2(\mathbb{R}^3)}$$

is the natural norm associated to (1.2). To achieve this, we associate to data u_0 in $L^2(\mathbb{R}^3)$ a decay character $r^* = r^*(u_0)$, which measures the "order" of $\widehat{u_0}(\xi)$ at $\xi = 0$ in frequency space and then find sharp decay estimates for solutions to the linear part

$$\partial_t \left(u - \alpha^2 \Delta u\right) - \nu \Delta u = 0 \tag{1.4}$$