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Parabolic partial differential equations with discrete state-dependent delay: Classical solutions and solution manifold

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Abstract

Classical solutions to PDEs with discrete state-dependent delay are studied. We prove the well-posedness in a set X_F which is analogous to the solution manifold used for ordinary differential equations with state-dependent delay. We prove that the evolution operators are C^1-smooth on the solution manifold.

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1. Introduction

Differential equations play an important role in describing mathematical models of many real-world processes. For many years the models are successfully used to study a number of physical, biological, chemical, control and other problems. A particular interest is in differential equations

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with many variables such as partial differential equations (PDE) and/or integral differential equations (IDE) in the case when one of the variables is time. Such equations are frequently called *evolution equations*. They received much attention from researchers from different fields since such equations could (in one way or another) discover future states of a model. It is generally known that taking into account the *past states* of the model, in addition to the present one, makes the model more realistic. This leads to the so-called delay differential equations (DDE). Historically, the theory of DDE was first initiated for the simplest case of ordinary differential equations (ODE) with constant delay (see the monographs [2,7,4,13] and references therein). Recently many important results have been extended to the case of delay PDEs with constant delay (see e.g., [26,6,25,28]).

Investigating the models described by DDEs it is clear that the constancy of delays is an extra assumption which significantly simplifies the study mathematically but is rarely met in the underlying real-world processes. The value of the delays can be time or state-dependent. Recent results showed that the theory of state-dependent delay equations (SDDE) essentially differs from the ones of constant and time-dependent delays. The basic results on ODEs with state-dependent delay can be found in [5,10,11,17,12,16,27] and the review [8]. The starting point of many mathematical studies is the well-posedness of an initial-value problem for a differential equation. It is directly connected with the choice of the space of initial functions (phase space). For DDEs with constant delay the natural phase space is the space of continuous functions. However, SDDEs non-uniqueness of solutions with continuous initial function has been observed in [5] for ODE case. The example in [5] was designed by choosing a non-Lipschitz initial function $\varphi \in C[-h,0]$ and a state-dependent delay such that the value $-r(\varphi) \in [-h,0]$ (at the initial function) is a non-Lipschitz point of φ. In order to overcome this difficulty, i.e., to guarantee unique solvability of initial value problems it was necessary to restrict the set of initial functions (and solutions) to a set of smoother functions. This approach includes the restrictions to layers in the space of Lipschitz functions, C^1 functions or the so-called solution manifold (a subset of $C^1[-h,0]$). As noted in [8, p. 465] "...typically, the IVP is uniquely solved for initial and other data which satisfy suitable Lipschitz conditions." The idea to investigate ODEs with state-dependent delays in the space of Lipschitz continuous functions is very fruitful, see e.g. [17,27]. In the present work we rely on the study of solution manifold for ODEs [14,16,27].

The study of PDEs with state-dependent delay is naturally more difficult and was initiated only recently [19–24]. In contrast to the ODEs with state-dependent delays, the possibility to exploit the space of Lipschitz continuous functions in the case of PDEs with state-dependent delays meets additional difficulties. One difficulty is that the solutions of PDEs usually do not belong to the space of Lipschitz continuous functions. Another difficulty is that the time-derivative of a solution belongs to a wider space comparing to the space to which the solution itself belongs. This fact makes the choice of the appropriate Lipschitz property more involved, and it depends on a particular model under consideration. It was already found (see [22] and [24]) that non-local operators could be very useful in such models and bring additional smoothness to the solutions. Further studies also show that approaches using C^1-spaces and solution manifolds (see [14,27] and [8] for ODE case) could also be used for PDE models, see [22,24]. In this work we combine the results for ODEs [8,16,27] and PDEs [22,24].

We also mention that a simple and natural additional property concerning the state-dependent delay which guarantees the uniqueness of solutions in the whole space of continuous functions was proposed in [21] and generalized in [23]. We will not develop this approach here.

Our goal in this paper is to investigate classical solutions to parabolic PDEs with discrete state-dependent delay. We find conditions for the well-posedness and prove the existence of a