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The Maslov and Morse indices for Schrödinger operators on [0, 1]

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Abstract

Assuming a symmetric potential and separated self-adjoint boundary conditions, we relate the Maslov and Morse indices for Schrödinger operators on [0, 1]. We find that the Morse index can be computed in terms of the Maslov index and two associated matrix eigenvalue problems. This provides an efficient way to compute the Morse index for such operators.

Keywords: Maslov index; Morse index; Schrödinger equation; Eigenvalues

1. Introduction

We consider eigenvalue problems

$$
\begin{aligned}
Hy &:= -y'' + V(x)y = \lambda y \\
\alpha_1 y(0) + \alpha_2 y'(0) &= 0 \\
\beta_1 y(1) + \beta_2 y'(1) &= 0,
\end{aligned}
\tag{1.1}
$$

where $y \in \mathbb{R}^n$, $V \in C([0,1])$ is a symmetric matrix in $\mathbb{R}^{n \times n}$, and α_1, α_2, β_1, and β_2 are real-valued $n \times n$ matrices such that

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$$\operatorname{rank}\begin{bmatrix}\alpha_1 & \alpha_2\end{bmatrix} = n; \qquad \operatorname{rank}\begin{bmatrix}\beta_1 & \beta_2\end{bmatrix} = n, \tag{1.2}$$

$$\alpha_1\alpha_2^t - \alpha_2\alpha_1^t = 0_{n\times n}; \qquad \beta_1\beta_2^t - \beta_2\beta_1^t = 0_{n\times n}, \tag{1.3}$$

where we use superscript t to denote matrix transpose, anticipating the use of superscript T to denote transpose in a complex Hilbert space described below. If (1.2)–(1.3) hold then without loss of generality we can take

$$\begin{aligned}\alpha_1\alpha_1^t + \alpha_2\alpha_2^t &= I,\\ \beta_1\beta_1^t + \beta_2\beta_2^t &= I\end{aligned} \tag{1.4}$$

(see, for example, [21, page 108]).

In particular, we are interested in counting the number of negative eigenvalues for H (i.e., the Morse index). We proceed by relating the Morse index to the Maslov index, which is described in Section 2. In essence, we'll find that the Morse index can be computed in terms of the Maslov index, and that while the Maslov index is less elementary than the Morse index, it's relatively straightforward to compute in the current setting.

The Maslov index has its origins in the work of V.P. Maslov [24] and subsequent development by V.I. Arnol'd [1]. It has now been studied extensively, both as a fundamental geometric quantity [3,11,15,25,26] and as a tool for counting the number of eigenvalues on specified intervals [4, 6–10,12,14,17–19]. In this latter context, there has been a strong resurgence of interest following the analysis by Deng and Jones (i.e., [12]) for multidimensional domains. Our aim in the current analysis is to rigorously develop a relationship between the Maslov index and the Morse index in the relatively simple setting of (1.1), and to take advantage of this setting to compute the Maslov index directly for example cases so that these properties can be illustrated and illuminated. Our approach is adapted from [10,12].

As a starting point, we define what we will mean by a *Lagrangian subspace*.

Definition 1.1. We say $\ell \subset \mathbb{R}^{2n}$ is a Lagrangian subspace if ℓ has dimension n and

$$(Jx, y)_{\mathbb{R}^{2n}} = 0,$$

for all $x, y \in \ell$. Here, $(\cdot,\cdot)_{\mathbb{R}^{2n}}$ denotes Euclidean inner product on $\mathbb{R}^{2n}$, and

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

with I_n the $n \times n$ identity matrix. We sometimes adopt standard notation for symplectic forms, $\omega(x, y) = (Jx, y)_{\mathbb{R}^{2n}}$.

A simple example, important for intuition, is the case $n = 1$, for which $(Jx, y)_{\mathbb{R}^2} = 0$ if and only if x and y are linearly dependent. In this case, we see that any line through the origin is a Lagrangian subspace of $\mathbb{R}^2$. As a foreshadowing of further discussion, we note that each such Lagrangian subspace can be identified with precisely two points on the unit circle S^1.

More generally, any Lagrangian subspace of $\mathbb{R}^{2n}$ can be spanned by a choice of n linearly independent vectors in $\mathbb{R}^{2n}$. We will generally find it convenient to collect these n vectors as the columns of a $2n \times n$ matrix $\mathbf{X}$, which we will refer to as a *frame* for ℓ.