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J. Differential Equations 260 (2016) 4550–4571

Journal of Differential Equations

www.elsevier.com/locate/jde

Hessian estimates in weighted Lebesgue spaces for fully nonlinear elliptic equations

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Received 29 August 2015; revised 18 November 2015

Available online 4 December 2015

Abstract

We prove global regularity in weighted Lebesgue spaces for the viscosity solutions to the Dirichlet problem for fully nonlinear elliptic equations. As a consequence, regularity in Morrey spaces of the Hessian is derived as well.

MSC: primary 35J60; secondary 35B45, 35D40, 35B65

Keywords: Fully nonlinear equation; Viscosity solution; Hessian estimates; Weighted Lebesgue space; Muckenhoupt weight; Morrey space

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http://dx.doi.org/10.1016/j.jde.2015.11.025

1. Introduction

The paper deals with the following Dirichlet problem for fully nonlinear elliptic equations

$$\begin{cases} F(D^2u, Du, u, x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Here, $F = F(X, z, s, x)$ is a real valued Carathéodory function defined on $S(n) \times \mathbb{R}^n \times \mathbb{R} \times \Omega$, where $S(n)$ is the set of $n \times n$ real symmetric matrices ordered in the usual way: $X \geq 0$ when $\langle X\xi, \xi\rangle \geq 0$ for all $\xi \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, and $Y \geq X$ means $Y - X \geq 0$. We assume that F is uniformly elliptic with ellipticity constants λ and Λ, that is, there exist constants λ and Λ with $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda \|Y\| \leq F(X + Y, z, s, x) - F(X, z, s, x) \leq \Lambda \|Y\|, \tag{1.2}$$

for all $X, Y \in S(n)$, $Y \geq 0$, $z \in \mathbb{R}^n$, $s \in \mathbb{R}$ and almost all $x \in \Omega$, and where $\|Y\| := \sup_{|x|=1} |Yx|$ that is equal to the maximum eigenvalue of Y whenever $Y \geq 0$.

Due to the discontinuous dependence on x of the nonlinear term F, the right notion of solution to the problem (1.1) would be that of function taken in a Sobolev space $W^{2,p}$ that satisfies the equation in a strong or viscosity sense and which vanishes identically on $\partial\Omega$. It was L. Caffarelli the first to derive in the seminal paper [2] interior *a priori* $W^{2,p}$-estimates for the solutions of (1.1) for all $p > n$, and these led to significant progress in the general study of fully nonlinear elliptic equations. By adapting the approach of Caffarelli, L. Wang developed in [25] the $W^{2,p}$-regularity theory of nonlinear parabolic equations. The restriction $p > n$ in [2] is due to the Aleksandrov–Bakel'man–Pucci maximum principle which turned out to be crucial in Caffarelli's approach. By using weak reverse Hölder inequalities, L. Escauriaza extended in [12] the results from [2] to the range $p > n - \epsilon$ with a small $\epsilon > 0$ depending on the ellipticity constants of the nonlinear operator considered. Recently, employing the techniques from [2] and [12], N. Winter derived in [26] boundary (and thus also global) $W^{2,p}$-*a priori* estimates for the solutions of (1.1), and proved $W^{2,p}$-solvability results as well. In the works cited, it is supposed that the nonlinear term F supports linear growths with respect to D^2u, Du and u (see (2.1) below), while its behavior in x is controlled in terms of small bounded mean oscillation (BMO) category. Just for the sake of completeness, let us note the papers [10,19] where $W^{2,n}$-solvability has been proved for Dirichlet and oblique derivative problems for quasilinear elliptic equations with quadratic gradient growths and where the discontinuity of the principal coefficients is measured in terms of vanishing mean oscillation (VMO). Further $W^{2,p}$-solvability results for certain uniformly elliptic fully nonlinear equations with superlinear or quadratic growths in the gradient have been obtained in [16,20].

The general aim of the present article is to extend the results of Winter [26] to the settings of weighted Sobolev spaces. More precisely, the functional framework we are dealing with is the space $W^{2,p}_w(\Omega)$, with a weight w taken in an appropriate Muckenhoupt class. Our goal is to prove that, under appropriate hypotheses on the data, for each $f \in L^p_w(\Omega)$ there exists a unique strong solution $u \in W^{2,p}_w(\Omega)$ of (1.1) that satisfies the estimate

$$\|u\|_{W^{2,p}_w(\Omega)} \leq c\|f\|_{L^p_w(\Omega)} \tag{1.3}$$