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Axisymmetric flow of ideal fluid moving in a narrow domain: A study of the axisymmetric hydrostatic Euler equations

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Abstract

In this article we will introduce a new model to describe the leading order behavior of an ideal and axisymmetric fluid moving in a very narrow domain. After providing a formal derivation of the model, we will prove the well-posedness and provide a rigorous mathematical justification for the formal derivation under a new sign condition. Finally, a blowup result regarding this model will be discussed as well.

MSC: 35Q31; 35Q35; 76B03

Keywords: Hydrostatic approximation; Formal derivation; Well-posedness; Formation of singularity; Local Rayleigh condition

Contents

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1. Introduction

In various applications in meteorology, oceanography, atmospheric dynamics, blood flow and pipeline transport, the vertical or radial length scale of the underlying flow is usually small compared to the horizontal length scale. To study these problems, the standard approach is to apply the hydrostatic approximations. For example, when a two-dimensional ideal fluid moves in a fixed and very narrow channel, one can describe the leading order behavior of the fluid motion by the two-dimensional hydrostatic Euler equations, which can be formally derived by the hydrostatic limit [1, § 4.6] or the least action principle [2]. Under the local Rayleigh condition, the formal derivation of the two-dimensional hydrostatic Euler equations via the hydrostatic limit was rigorously justified in [3–5]. Without the local Rayleigh condition, the formal derivation may not be valid [3,6]. The local-in-time existence and uniqueness under the analyticity assumption [7], the local Rayleigh condition [8,5], or their combinations in different regions [9] are also known, but the global-in-time existence is still open. Furthermore, for a general initial data, the two-dimensional hydrostatic Euler equations are somewhat ill-posed: see [10] for the linearized instability, and [11,12] for the formation of singularities.

In this paper, we study the leading order behavior of axisymmetric and ideal flows moving in a very narrow domain in three spatial dimensions. The prime objectives of this paper are as follows:

(i) to formally derive the axisymmetric hydrostatic Euler equations, which describe the leading order behavior of axisymmetric Euler flows moving in a thin tube, via the hydrostatic limit (see Subsection 2.1);
(ii) to introduce a *new* sign condition (see inequality (2.8) below), which is an analogue of the local Rayleigh condition in two spatial dimensions, for the axisymmetric hydrostatic Euler equations in three spatial dimensions;
(iii) to prove the well-posedness of the axisymmetric hydrostatic Euler equations under the new sign condition (see Theorem 2.2, Sections 3 and 4, as well as Appendix C);
(iv) to provide a rigorous mathematical justification of the formal derivation for the axisymmetric hydrostatic Euler equations under the new sign condition (see Theorem 2.4 and Section 5);