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Global classical solutions to the Cauchy problem of conservation laws with degenerate diffusion

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Abstract

In this paper we prove the existence of the unique global classical solution with small initial data to the Cauchy problem of a scalar conservation law with degenerate diffusion by establishing the uniform *a priori* decay estimates of solutions. In order to compensate the degeneracy on the x_1 direction by the diffusion on other directions, we introduce the frequency decomposition method and obtain the low frequency estimate and the high frequency estimate of the solution by the Green's function method and energy method respectively.

Keywords: Classical solution; Global existence; Conservation law; Degenerate diffusion; Frequency decomposition method; Green's function method

1. Introduction

In this paper, we are concerned with the global well-posedness of classical solutions to the following Cauchy problem of conservation law with degenerate diffusion:

$$\begin{cases} u_t - \Delta_{x'} u = \nabla \cdot f(u), \ x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

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Here $f(u) = (f_1(u), \cdots, f_n(u))$ is a given vector function of $u \in \mathbb{R}$, where $f_i(u) = O(|u|^{1+\theta})$ $(i = 1, \cdots, n)$ are sufficiently smooth and $\theta \geq 1$ is an integer. $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})$ is the gradient operator and then $\nabla\cdot$ is the divergence operator. We adopt the notation $\Delta_{x'} := \sum_{i=2}^{n} \partial_{x_i}^2$ to denote the Laplacian operator with respect to $x' = (x_2, \cdots, x_n)$. We also denote $g(u) = (f_2(u), \cdots, f_n(u))$ for convenience, then $f(u) = (f_1(u), g(u))$.

More generally, the equation

$$u_t - \nabla \cdot (A(u)\nabla u)) = \nabla \cdot f(u) \tag{1.2}$$

is called a degenerate parabolic–hyperbolic equation because it is a coupling of the degenerate diffusion term $\nabla \cdot (A(u)\nabla u)$ (where the diffusion matrix $A(u)$ is just non-negative instead of positive) and the convection term $\nabla \cdot f(u)$. This equation arises from many physical phenomena, such as two phase flows in porous medium [11] and sedimentation-consolidation process [3]. Due to the degeneracy, most literatures are about various kinds of weak solutions (see [2,4–7,9, 10,12–14,16–18] and the references cited therein).

The equation in (1.1) is a special quasi-isotropic case of (1.2) with $A(u) = \text{diag}(0, 1, \cdots, 1)$. The degeneracy of $A(u)$ in the x_1 direction results in the lacking of viscous effect in the x_1 direction, which is the main difficulty of this equation since the commonly used parabolic methods are not applicable here. For example, when deriving bounded estimates of solutions by energy method we find that some nonlinear terms involving derivatives with respect to x_1 are out of control. In [8] we considered the initial–boundary value problem of (1.1) on a special domain $\mathbb{R} \times \prod_{i=2}^{n}(0, L_i)$ and established the global existence and exponential decay estimates to the solution with small initial data, which depends strongly on a Poincaré-like inequality which holds due to the boundedness of the domain along x' direction and the homogeneous boundary conditions. In this paper we are interested in the existence and decay estimates of global classical solution to the Cauchy problem (1.1) with small initial data. To prove the global existence of classical solution to Cauchy problem (1.1), we employ the standard continuity argument, namely, we obtain the local solution first and then extend it to a global-in-time solution by establishing the uniform estimates of the solution.

Compared with [8], we do not have a general Poincaré-like inequality, we only have a Poincaré-like inequality for high frequency part, so we need to figure out a new approach based on this fact. Our main idea is making full use of the decay produced by diffusion in the x' direction. The key point is to overcome the difficulty caused by the degeneracy in the x_1 direction by introducing the frequency decomposition method and combining bounded estimates with decay estimates tactfully. More precisely, we can carry out our uniform estimates in the following steps:

1. The *a priori* decay assumption: we first set the *a priori* assumption with optimal algebraic decay rate as $t \to +\infty$ for both L^2 and L^∞ norms of the solution. Here, optimal rate means that the decay rate of solution is consistent with the decay rate of Green's function to problem (1.1).
2. Bounded estimates: under the *a priori* assumption we establish the bounded estimates of solutions for any order derivatives by using the energy method and treating the derivatives along the x_1 direction and the x' direction respectively.
3. Decay estimates: based on the bounded estimates we introduce the frequency decomposition method to close our *a priori* decay estimates. For the low frequency part we use the Green's function method, and for the high frequency part we use the energy estimates along with the Poincaré-like inequality with respect to the x' direction.