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A class of Hamilton–Jacobi equations with constraint: Uniqueness and constructive approach

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Abstract

We discuss a class of time-dependent Hamilton–Jacobi equations, where an unknown function of time is intended to keep the maximum of the solution to the constant value 0. Our main result is that the full problem has a unique viscosity solution, which is in fact classical. The motivation is a selection–mutation model which, in the limit of small diffusion, exhibits concentration on the zero level set of the solution of the Hamilton–Jacobi equation.

Uniqueness is obtained by noticing that, as a consequence of the dynamic programming principle, the solution of the Hamilton–Jacobi equation is classical. It is then possible to write an ODE for the maximum of the solution, and treat the full problem as a nonstandard Cauchy problem.

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1. Introduction

1.1. Model and question

The purpose of this paper is to discuss existence and uniqueness for the following problem, with unknowns $(I(t), u(t,x))$:

$$\begin{cases} \partial_t u = |\nabla u|^2 + R(x, I) \ (t > 0, x \in \mathbb{R}^d), \quad \max_x u(t,x) = 0 \\ I(0) = I_0 > 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $I_0 > 0$ and $u_0(\cdot)$ and $R(\cdot, I)$ are concave functions with quadratic decrease and $\max_x u_0(x) = 0$. For a given continuous function $I(t)$, $u(t,x)$ solves a Hamilton–Jacobi equation. The unknown I may be thought of as a sort of regulator, or a sort of Lagrange multiplier, to maintain the maximum of u equal to 0. The constraint on the maximum of $u(t,.)$ makes the problem nonstandard.

Existence of a solution (u, I) to (1) is not new, the first result is due to Perthame and Barles [21] (see also Barles–Perthame–Mirrahimi [3] for a result with weaker assumptions). An important improvement is given by Lorz, Perthame and the first author in [16]; they indeed notice that a concavity assumptions on R – that we also make here – entail regularity. This allows them to derive the dynamics of the maximum point of a solution $u(t,x)$. See also [17]. Both types of results rely on a special viscous approximation of (1) – see equation (4) below. Uniqueness, however, has remained an open problem, apart from a very particular case [21].

The main goal of the paper is to prove the missing uniqueness property; a result that we had already announced in [19]. We also provide a constructive existence proof which was not available in the previous existence results [21,3,16]. Two important consequences, that we will present in a forthcoming paper [20] (see also [19]), will be the convergence of the underlying selection–mutation model in a stronger sense than what is known, and asymptotic expansion of the viscous solution. The asymptotic expansion, which allows to approximate the phenotypical distribution of the population when the mutation steps are small but nonzero, is particularly interesting in view of biological applications. One of the main ingredients will be regularity under suitable concavity assumptions on R and u_0, which is far from being available in general. Instead of relying on viscous approximations we will prove these results directly for the equation

$$u_t = |\nabla u|^2 + R(t,x). \tag{2}$$

This will allow a much easier treatment than in the usual viscosity sense. The uniqueness result will also be helpful to develop the so-called Hamilton–Jacobi approach (see for instance Diekmann et al. [9,21,16] and Subsection 1.2) to study more complex models describing selection and mutations. For instance, our result would allow to generalize a result due to Perthame and the first author in [18] on a selection model with spatial structure, where the proof relies on the uniqueness of the solution to a corresponding Hamilton–Jacobi equation with constraint.