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Critical non-Sobolev regularity for continuity equations with rough velocity fields

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Abstract

We present a divergence free vector field in the Sobolev space H^1 such that the flow associated to the field does not belong to any Sobolev space. The vector field is deterministic but constructed as the realization of a random field combining simple elements. This construction illustrates the optimality of recent quantitative regularity estimates as it gives a straightforward example of a well-posed flow which has nevertheless only very weak regularity.

MSC: 34C11; 35L45; 37C10; 34A12; 34A36

1. Introduction

Given a vector field $u(t,x) : \mathbb{R}_+ \times \mathbb{R}^d$, we consider the flow $X(t,s,x)$ associated to it

$$\partial_t X(t,s,x) = u(t, X(t,s,x)), \quad X(t=s,s,x) = x. \tag{1.1}$$

The well-known Cauchy–Lipschitz theorem implies that if u is locally Lipschitz, then Eq. (1.1) is locally well posed for every choice of x. The flow X is in that case also locally Lipschitz in x and this regularity is intimately connected with the existence and uniqueness result.

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Well posedness for *non-Lipschitz* velocity fields u was first obtained by R.J. DiPerna and P.-L. Lions in [18] with the introduction of renormalized solutions for the associated continuity equation

$$\partial_t \rho + \operatorname{div}(\rho\, u) = 0, \quad \rho(t=0,x) = \rho^0(x). \tag{1.2}$$

The result [18] required $u \in L^1_t W^{1,1}_x$ with appropriate bounds on $\operatorname{div} u$.

The renormalization property of weak solutions to (1.2) was proved for only $u \in BV$ and $\operatorname{div} u \in L^p$ for some $p \geq 1$ in the seminal article [3] (see [7] for a first result in the kinetic context). This is an important step as the regularity $u \in BV$ is critical for several problems.

The assumption on $\operatorname{div} u$ can be relaxed in some cases by assuming instead that there exists a solution to (1.2) which is bounded from below and from above. We refer to [5], though one then typically requires $u \in SBV$ (see also [22]).

The optimality of $u \in BV$ for such a general setting was demonstrated in [17] with an example of almost BV vector field leading to non-unique flows. We refer the readers to the two excellent summaries in [16] and a more recent [4].

Contrary to the classical Cauchy–Lipschitz theory, well posedness results that rely on the theory of renormalized solutions do not in general provide explicit regularity estimates on the flow X. In contrast a purely quantitative approach was introduced in [15], focusing on estimates like

$$\int_{\mathbb{R}^d} \fint_{B(x,h)} \log\left(1 + \frac{|X(1,0,x) - X(1,0,y)|}{h}\right) dy\, dx \leq C_d\, \|u\|_{L^1_t W^{1,p}_x}, \tag{1.3}$$

where $\fint_\Omega$ denotes the average over the domain of integration Ω. Such direct estimates on the flow give both quantitative regularity results and well posedness (see also [20,24,25] for another approach of well posedness working directly with the flow).

A regularity result like (1.3) also implies some regularity on the solution to the continuity equation (1.2). Regularity estimates have been obtained directly from Eq. (1.2) in [6] and in the more general context of non-linear continuity equations but they require more complicated tools.

The regularity given by (1.3) is quite weak, a sort of log of a derivative and very far from the Lipschitz regularity obtained if $u \in W^{1,\infty}_{loc}$. It is therefore a very natural question whether this regularity is optimal or if something better should be expected.

It is relatively easy to construct examples of flows with weak regularity with just the requirement that $u \in L^2_t H^1_x$ but no assumption on $\operatorname{div} u$. But there are very few examples if one imposes that $\operatorname{div} u \in L^\infty$ or even that *u be divergence free*. To the author's knowledge, the only such other example was obtained in [2]; other interesting examples of velocity fields occur in the slightly different but related context of mixing (see below).

Note that the assumption $\operatorname{div} u = 0$ is an especially strong constraint in low dimensions: Obviously in dimension 1 it implies that u is constant and then $\rho(t,.)$ is as smooth as ρ^0 so that (1.4) cannot stand.

This article presents such an example of weak regularity in the minimum dimension where it is possible, namely $d = 2$.