Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 4758–4785

Journal of Differential Equations

www.elsevier.com/locate/jde

Indefinite fractional elliptic problem and Liouville theorems

Wenxiong Chen [a,b,*,1], Jiuyi Zhu [c,2]

[a] College of Mathematics and Information Science, Henan Normal University, China
[b] Department of Mathematical Sciences, Yeshiva University, New York, NY 10033, USA
[c] Department of Mathematics, Johns Hopkins University, Baltimore, MD 21218, USA

Received 21 April 2014; revised 24 August 2015

Available online 8 December 2015

Abstract

In this paper, we consider an indefinite fractional elliptic problem. A corresponding Liouville-type theorem is established. Furthermore, we obtain a priori estimate for solutions in a bounded domain by blowing-up and rescaling. We also classify the solutions of some degenerate elliptic equation originated from the fractional Laplacian.

MSC: 35B53; 35B45; 35B44; 35J70

Keywords: The fractional Laplacian; Indefinite problem; Blowing up and rescaling; A priori estimates; Liouville theorem; Classification

1. Introduction

The fractional Laplacian has attracted much attention recently. It has applications in mathematical physics, biological modeling, mathematical finances, and so on. Especially, it appears in turbulence and water wave, anomalous dynamics, flames propagation, chemical reactions in

* Corresponding author at: Department of Mathematical Sciences, Yeshiva University, New York, NY 10033, USA.
E-mail addresses: wchen@yu.edu (W. Chen), jzhu43@math.jhu.edu (J. Zhu).

1 The first author is partially supported by the Simons Foundation Collaboration Grant for Mathematicians 245486.
2 The second author is partially supported by the National Science Foundation grant DMS 1500468.

http://dx.doi.org/10.1016/j.jde.2015.11.029
0022-0396/

liquids, population dynamics, geophysical fluid dynamics, and American options in finance. It also has connections to conformal geometry, e.g. [6].

The fractional Laplacian in $\mathbb{R}^n$ is a pseudo differential operator defined as

$$(-\triangle)^{\frac{\alpha}{2}} u = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy, \tag{1.1}$$

where $P.V.$ stands for the Cauchy principle value. Another equivalent definition is given by the Fourier transform, that is,

$$\widehat{(-\triangle)^{\frac{\alpha}{2}} u}(\xi) = |\xi|^{\alpha} \hat{u}(\xi),$$

for functions $u(x)$ in the Schwartz class. Observe that the above definitions are nonlocal. To circumvent this, Caffarelli and Silvestre [8] introduced a local realization through the Dirichlet–Neumann map of an appropriate degenerate elliptic operator in $\mathbb{R}^{n+1}_+$. Based on this extension, we are able to study the fractional Laplacian $(-\triangle)^{\frac{\alpha}{2}}$ in a local way and to use the existing tools for semilinear elliptic equations. More precisely, if $u \in H^{\frac{\alpha}{2}}(\mathbb{R}^n)$, then w is its extension in $\mathbb{R}^{n+1}_+$, if it solves the equation

$$\begin{aligned} -div(y^{1-\alpha} \nabla w) &= 0 && \text{in } \mathbb{R}^{n+1}_+, \\ w &= u && \text{on } \mathbb{R}^n \times \{y = 0\}. \end{aligned}$$

It is shown in [8] that

$$-\lim_{y \to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y}(x, y) = k_\alpha (-\triangle)^{\frac{\alpha}{2}} u(x),$$

where the constant

$$k_\alpha = \frac{2^{1-\alpha} \Gamma(1 - \frac{1}{2}\alpha)}{\Gamma(\frac{1}{2}\alpha)}.$$

For a bounded domain, there are several ways to define fractional Laplacians, which coincide when the domain is the entire Euclidean space, but can otherwise be quite different. Among those, one is still defined by integral (1.1), and another one is defined by the eigenvalues of the Laplacian in the same domain as introduced in [9] for the case $\alpha = 1$ and in [3] for its generalization to $0 < \alpha < 2$.

Let $\{\lambda_k, \phi_k\}_{k=1}^{\infty}$ be the eigenvalues and corresponding eigenfunctions of the Laplace operator $-\triangle$ in Ω with zero Dirichlet boundary values:

$$\begin{cases} -\triangle \phi_k = \lambda_k \phi_k & \text{in } \Omega, \\ \phi_k = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

such that $\|\phi_k\|_{L^2(\Omega)} = 1$ and $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots$.