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J. Differential Equations 260 (2016) 4786–4804

Journal of Differential Equations

www.elsevier.com/locate/jde

Increasing stability in the inverse source problem with many frequencies

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Received 3 December 2014; revised 7 April 2015

Available online 4 December 2015

Abstract

We study increasing stability in the interior inverse source problem for the Helmholtz equation from boundary Cauchy data for multiple wave numbers. By using the Fourier transform with respect to the wave number, explicit bounds for the analytic continuation, uniqueness of the continuation results, and exact observability bounds for the wave equation, a sharp uniqueness result and an increasing (with larger wave numbers intervals) stability estimate are obtained. Numerical examples in 3 spatial dimension support the theoretical prediction.

MSC: 35R30; 78A46

Keywords: Increasing stability; Inverse source problems; Multiple frequencies

1. Introduction

We are interested in uniqueness and stability in the inverse source problems for elliptic equations when the source term, supported in a bounded domain Ω, is to be found from full/partial boundary data on $\partial\Omega$. One important example is the recovery of acoustic sources from boundary

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http://dx.doi.org/10.1016/j.jde.2015.11.030
0022-0396/

measurements of the pressure. This type of inverse source problems is also motivated by applications in antenna synthesis [1], biomedical imaging [2] and various kinds of tomography [3,4]. From the boundary data for one single differential equation, it is not possible to find the source uniquely [5, Ch. 4], but in case of a family of equations (like the Helmholtz equation for various wave numbers in $(0, K)$) one can regain uniqueness. Then the crucial issue for applications is the stability of source recovery. In general, a feature of inverse problems for elliptic equations is a logarithmic type stability estimate which results in stable recovery of only few parameters describing the source and hence yields very low resolution numerically. For the Helmholtz equation we will show uniqueness of the source from the data on any open non-empty part of the boundary and for arbitrary positive wave number K and increasing stability when the data are given on the whole boundary and K is getting large.

We are aware only on uniqueness and numerical results of [6] and the first increasing stability results [7,8] obtained in more particular case (in both two and three dimensions) and by quite different (direct spatial Fourier analysis) methods. These estimates do not look like ours. Available Lipschitz stability estimates either assume that the source term is contained in a finite-dimensional space (or manifold) (with constants depending on k and the dimension of this space) or that the data are given for all wave numbers k, when one immediately uses known results for hyperbolic equations. In the current work we also use the Fourier transform in time to reduce our inverse source problem to identification of the initial data in the hyperbolic initial value problem by lateral Cauchy data (observability in control theory). We derive our increasing stability estimates by obtaining sharp bounds of the analytic continuation of the data from $(0, K)$ onto $(0, +\infty)$ and combining them with known optimal Lipschitz stability for the wave equation (exact observability inequalities). We note that our current analysis is in 3D and is transparent due to use of the Huygens' principle [9].

More is known about uniqueness and increasing stability in the Cauchy problem for elliptic equations and for identification of the Schrödinger potential. Classic Carleman estimates imply some conditional Hölder type stability estimates for solutions of the elliptic Cauchy problem. In 1960 [10] F. John showed that for the continuation of the Helmholtz equation from the unit disk onto any larger disk the stability estimate, which is uniform with respect to the wave numbers, is still of logarithmic type. In recent papers [11–15] it was demonstrated that in a certain sense stability is always improving for larger k under (pseudo) convexity conditions on the geometry of the domain and of the coefficients of the elliptic equation. Increasing stability for the Schrödinger potential was demonstrated in [16,17].

This paper is organized as follows. In Section 2 we prove uniqueness in the inverse source problem for the Helmholtz equation. Here we use analyticity of the solution $u(,k)$ with respect to k and the Holmgren–John sharp uniqueness of the continuation theorem. This proof can be easily extended to a general elliptic operator. In Section 3 we obtain explicit bounds of the harmonic measure of the interval $(0, K)$ in a sector of the complex plane $k = k_1 + ik_2$ and use them to derive explicit bounds of the analytic continuation of $u(,k)$ from $(0, K)$ onto the real axis. These results are crucial for the proof of increasing stability estimate in Section 4. Finally numerical examples in Section 5 shed light on the theoretical prediction and confirm the increasing stability from multiple frequencies.

2. Uniqueness of source identification

Throughout the content, C denote generic constants depending on the domain Ω. $\|u\|_{(l)}(\Omega)$ denotes the standard norm in the Sobolev space $H^l(\Omega)$.