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A free boundary problem arising from a stochastic optimal control model under controllable risk ☆

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Abstract

We consider a Barenblatt parabolic equation

$$v_t - \sup_{0 \le a \le 1} \left\{ \frac{1}{2} a^2 \sigma^2 v_{xx} + a\mu v_x - cv + x \right\} = 0.$$

This equation comes from finance. In our model, the risk of the insurance company is controllable. The so-called proportional reinsurance means that it is possible for the cedent to divert $1 - a$ fraction of all premiums to the reinsurance company with the obligation from the latter to pay $1 - a$ fraction of each claim. The insurance company is willing to maximize the expected dividends (value function) by choosing the control function $a(t)$. First, we proved v is an increasing and convex function with respect to x by the method of stochastic analysis. We can see from the equation above that if $-\frac{\mu}{\sigma^2}\frac{v_x}{v_{xx}} < 1$, then the optimal strategy $a^*(x,t) = -\frac{\mu}{\sigma^2}\frac{v_x}{v_{xx}} < 1$, in this situation, the optimal fraction which must be reinsured is $1 - a^*$. Otherwise, if $-\frac{\mu}{\sigma^2}\frac{v_x}{v_{xx}} \ge 1$ or $v_{xx} = 0$, $a^* = 1$, in this situation, it is optimal to take the maximal risk, using no reinsurance. Thus, we divide the domain into two parts, diverting region $\mathcal{D}$ and non-diverting region $\mathcal{ND}$. In these two regions, $v(x,t)$ satisfies different types of second-order partial differential equations, the former is fully nonlinear equation, and the latter is a linear equation. The junction of the two regions, i.e., free boundary has particular financial implications. We prove that it can be expressed as a functional form $x = h(t)$. In this paper we not only prove there exists an unique solution v which is three times differentiable, but we

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also prove $x = h(t)$ is a differentiable curve, and we show its upper and lower bounds and starting point $h(0)$.

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Keywords: Parabolic Barenblatt equation; Free boundary; Stochastic optimal control; Insurance company; Reinsurance

1. Introduction

Barenblatt equation has a wide range of applications in many areas, such as mechanical engineering (see [9]), and finance (see [3,7,10,11,18,19,4]). Some literatures have very good results about Barenblatt equation. In [12], Krylov proved this equation for bounded domain has classical solution under general circumstances.

In finance, Barenblatt equations arise from stochastic optimal control models which are called HJB equations. Our problem comes from dividend optimization/risk control models (see [17]), which are famous as well as consumption/investment models, there are many problems and related references such as [1,8,2], but very few of them consider finite-time models. Barenblatt equation usually corresponds to some PDEs in different regions (usually it corresponds to the number of control variables), and the solutions of the PDEs are governed by the so-called "principle of smooth fit". For the case of one-dimensional ODE, which corresponds to an infinite maturity model with one risk asset, the general solutions in each region could be expressed with some unknown constant parameters, and then by the boundary conditions of fixed boundary points and using the "principle of smooth fit" at the free boundary points, the unknown parameters could be determined (such as [17,1]). But the solution to the corresponding parabolic equation arise from finite maturity model, could not be explicitly solved, thus, qualitatively explore the properties of the solution and the free boundary becomes necessary. The free boundary often represents the dividing curve between using two different policies in finance, therefore, properties of the free boundary play an important role in financial decision-making. The goal of this paper is by using the PDE's techniques to determine the value function of the control problem, to examine how smooth it is, and to characterize the optimal policy.

In this paper, we consider a particular initial boundary value problem of Barenblatt parabolic equations,

$$\begin{cases} v_t - \sup\limits_{0\le a\le 1}\left\{\frac{1}{2}a^2\sigma^2 v_{xx} + a\mu v_x - cv + x\right\} = 0, & (x,t)\in(0,\infty)\times(0,T], \\ v(0,t) = 0, \\ v(x,0) = 0, \end{cases} \tag{1.1}$$

where μ, σ, c are positive constants, which represent the rate of yields, the volatility of risk assets, the interest rate of risk-free assets. In the infinite maturity model, namely ODE form, the literature [17] has given the expression of the solution and the free boundary point, which provides a reference for us.