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Uniqueness of the interior transmission problem with partial information on the potential and eigenvalues

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Abstract

The inverse spectral problem of determining a spherically symmetric wave speed v is considered in a bounded spherical region of radius b. A uniqueness theorem for the potential q of the derived Sturm–Liouville problem $B(q)$ is presented from the data involving fractions of the eigenvalues of the problem $B(q)$ on a finite interval and knowledge of q over a corresponding fraction of the interval. The methods employed rest on Weyl-function techniques and properties of zeros of a class of entire functions.

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Keywords: Interior transmission problem; Inverse spectral problem; Transmission eigenvalue; Weyl-function

1. Introduction

The interior transmission problem is a non-selfadjoint boundary-value problem for a pair of fields Ψ and Ψ_0 in a bounded and simply connected domain Ω of $\mathbb{R}^n$ with the sufficiently smooth boundary $\partial\Omega$. It was first stated in [10] and can be formulated [8,10,12,14] as

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$$\begin{cases} \Delta\Psi + k^2 n(x)\Psi = 0, \ x \in \Omega, \\ \Delta\Psi_0 + k^2\Psi_0 = 0, \ x \in \Omega, \\ \Psi = \Psi_0, \ \frac{\partial\Psi}{\partial\mathbf{n}} = \frac{\partial\Psi_0}{\partial\mathbf{n}}, \ x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Δ denotes the Laplacian, k^2 is the spectral parameter, $\mathbf{n}$ represents the outward unit normal to the boundary $\partial\Omega$, and the positive quantity $n(x)$ corresponds to the square of the refractive index of the medium at location x in the electromagnetic case or the reciprocal of the square of the sound speed $v(x)$ in the acoustic case, i.e. $v(x) := \frac{1}{\sqrt{n(x)}}$. In the acoustic case, $\sqrt{n(x)}$ is usually called the slowness. Without loss of generality, we can assume that in the region exterior to Ω, the speed of the electromagnetic wave is 1 or the sound speed is 1 in the acoustic case. This boundary value problem is called the interior transmission problem.

In the case $n = 3$, where $\Omega = \Omega_b$ is a ball of radius $b > 0$ centered at the origin and $n(x)$ is spherically symmetric ($n(x) = n(r)$, $r = |x|$), the boundary value problem (1.1) becomes equivalent to a nonstandard Sturm–Liouville-type eigenvalue problem with the spectral parameter appearing in the boundary condition at the right endpoint. Our assumptions on $n(r)$ are that $n(r)$ is positive and $n(r) \in W_2^2[0, b]$, $n(b) = 1$, $n'(b) = 0$. In this paper we consider the inverse spectral problem of recovering the function $n(r)$ from the so-called transmission eigenvalues for which the corresponding eigenfunctions are spherically symmetric.

Under the above assumptions this inverse spectral problem is equivalent to recovering the potential $q(x)$ from the spectrum of the following boundary value problem

$$B(q): \begin{cases} y''(x) + (\lambda - q(x))y(x) = 0, \quad 0 < x < 1, \\ y(0) = 0 = y(1)\cos(\sqrt{\lambda}a) - y'(1)\dfrac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}}. \end{cases} \tag{1.2}$$

Here $\sqrt{\lambda}$ is the square root branch with $\mathrm{Im}(\sqrt{\lambda}) \geq 0$ and

$$q(x) = B^2\Big(\frac{n''(r)}{4n^2(r)} - \frac{5(n'(r))^2}{16n^3(r)}\Big), \quad x = \frac{1}{B}\int_0^r \sqrt{n(\zeta)}d\zeta,$$

$$\lambda = B^2k^2, \quad a = \frac{b}{B}, \quad B = \int_0^b \sqrt{n(\zeta)}d\zeta.$$

Denote $\tilde{n}(x) := n(r)$. Then the function $\sqrt[4]{\tilde{n}(x)}$ satisfies the following Cauchy problem:

$$(\sqrt[4]{\tilde{n}(x)})'' = q(x)\sqrt[4]{\tilde{n}(x)}, \quad 0 < x < 1,$$
$$\sqrt[4]{\tilde{n}}(1) - 1 = 0 = (\sqrt[4]{\tilde{n}})'(1).$$

Thus, $q(x)$ uniquely determines $\tilde{n}(x)$, $0 < x < 1$. Again, from $x = \frac{1}{B}\int_0^r \sqrt{n(\zeta)}d\zeta$ we get

$$\frac{dr}{dx} = \frac{B}{\sqrt{\tilde{n}(x)}}$$