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J. Differential Equations 260 (2016) 4918–4972

Journal of Differential Equations

www.elsevier.com/locate/jde

The criticality of centers of potential systems at the outer boundary ☆

F. Mañosas [a], D. Rojas [a,*], J. Villadelprat [b]

[a] *Departament de Matemàtiques, Universitat Autònoma de Barcelona, Barcelona, Spain*
[b] *Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Tarragona, Spain*

Received 12 June 2015; revised 13 November 2015

Available online 15 December 2015

Abstract

The number of critical periodic orbits that bifurcate from the outer boundary of a potential center is studied. We call this number the criticality at the outer boundary. Our main results provide sufficient conditions in order to ensure that this number is exactly 0 and 1. We apply them to study the bifurcation diagram of the period function of $X = -y\partial_x + ((x+1)^p - (x+1)^q)\partial_y$ with $q < p$. This family was previously studied for $q = 1$ by Y. Miyamoto and K. Yagasaki.

MSC: 34C07; 34C23; 37C25

Keywords: Center; Period function; Critical periodic orbit; Bifurcation; Criticality

1. Introduction and setting of the problem

In this paper we study planar differential systems

$$\begin{cases} \dot{x} = f(x, y), \\ \dot{y} = g(x, y), \end{cases}$$

☆ All the authors are partially supported by the MEC/FEDER grant MTM2014-52209-C2-1-P.
* Corresponding author.
E-mail addresses: manyosas@mat.uab.cat (F. Mañosas), rojas@mat.uab.cat (D. Rojas), Jordi.Villadelprat@urv.cat (J. Villadelprat).

http://dx.doi.org/10.1016/j.jde.2015.11.040

where f and g are analytic functions on some open subset U of $\mathbb{R}^2$. A singular point $p \in U$ of the vector field $X = f(x, y)\partial_x + g(x, y)\partial_y$ is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The largest punctured neighbourhood with this property is called the *period annulus* of the center and it will be denoted by $\mathscr{P}$. Henceforth $\partial\mathscr{P}$ will denote the boundary of $\mathscr{P}$ after embedding it into $\mathbb{RP}^2$. Clearly the center p belongs to $\partial\mathscr{P}$, and in what follows we will call it the *inner boundary* of the period annulus. We also define the *outer boundary* of the period annulus to be $\Pi := \partial\mathscr{P} \setminus \{p\}$. Note that Π is a non-empty compact subset of $\mathbb{RP}^2$. The *period function* of the center assigns to each periodic orbit in $\mathscr{P}$ its period. If the period function is constant, then the center is said to be *isochronous*. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parametrize this set. This can be done by taking an analytic transverse section to X on $\mathscr{P}$, for instance an orbit of the orthogonal vector field $X^{\perp}$. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization, then $s \longmapsto T(s) := \{\text{period of } \gamma_s\}$ is an analytic map that provides the qualitative properties of the period function that we are concerned about, in particular the existence of *critical periods*, which are isolated critical points of this function, i.e. $\hat{s} \in (0, 1)$ such that $T'(s) = \alpha(s - \hat{s})^k + \mathrm{o}\big((s - \hat{s})^k\big)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center. One can readily see that this definition does not depend on the particular parametrization of the set of periodic orbits used. Critical periodic orbits in the study of the period function play an equivalent role to limit cycles, which is a fundamental notion in qualitative theory of differential systems in the plane.

Suppose now that the vector field X depends on a parameter $\mu \in \Lambda$, where Λ is an open set of $\mathbb{R}^d$. Thus, for each $\mu \in \Lambda$, we have an analytic vector field X_μ, defined on some open subset U_μ of $\mathbb{R}^2$, with a center at p_μ. Concerning the regularity with respect to the parameter, we shall assume that $\{X_\mu\}_{\mu\in\Lambda}$ is a continuous family of planar vector fields, meaning that the map $(x, y, \mu) \longmapsto X_\mu(x, y)$ is continuous on the subset $\{(x, y, \mu); \mu \in \Lambda \text{ and } (x, y) \in U_\mu\}$ of $\mathbb{R}^{d+2}$. Fix $\hat{\mu} \in \Lambda$ and, following the notation introduced previously, let $\Pi_{\hat{\mu}}$ be the outer boundary of the period annulus $\mathscr{P}_{\hat{\mu}}$ of the center at $p_{\hat{\mu}}$ of $X_{\hat{\mu}}$. The aim of the present paper is to provide tools in order to study the following bifurcation problem: which is the number of critical periodic orbits that can emerge or disappear from $\Pi_{\hat{\mu}}$ as we move slightly the parameter $\mu \approx \hat{\mu}$? We shall call this number the criticality of the outer boundary of the period annulus. In order to define it precisely we adapt the notion of cyclicity (cf. [1,2]), which is its counterpart in the study of limit cycles.

Definition 1.1. We define the *criticality* of the pair $(\Pi_{\hat{\mu}}, X_{\hat{\mu}})$ with respect to the deformation X_μ to be $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) := \inf_{\delta,\varepsilon} N(\delta, \varepsilon)$, where

$$N(\delta, \varepsilon) = \sup\left\{\text{number of critical periodic orbits } \gamma \text{ of } X_\mu\text{: } d_H(\gamma, \Pi_{\hat{\mu}}) \leqslant \varepsilon \text{ and } \|\mu - \hat{\mu}\| \leqslant \delta\right\},$$

with d_H being the Hausdorff distance between compact sets of $\mathbb{RP}^2$. $\square$

In other words, what we call the criticality $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big)$ is the maximal number of critical periodic orbits that tend to $\Pi_{\hat{\mu}}$ in the Hausdorff topology of the non-empty compact subsets of $\mathbb{RP}^2$ as $\mu \to \hat{\mu}$.

Definition 1.2. We say that $\hat{\mu} \in \Lambda$ is a *local regular value of the period function at the outer boundary* of the period annulus if $\mathrm{Crit}\big((\Pi_{\hat{\mu}}, X_{\hat{\mu}}), X_\mu\big) = 0$. Otherwise we say that it is a *local bifurcation value of the period function at the outer boundary.* $\square$