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Pullback attractors of the Jeffreys–Oldroyd equations ☆

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Abstract

We study the dynamics of the Jeffreys–Oldroyd equation using the theory of trajectory pullback attractors. We prove an existence theorem for weak solutions and use it to construct a family of trajectory spaces and to specify the class of attracted families of sets, which includes families bounded in the past. Finally, we prove the existence of the trajectory and global pullback attractors of the model.

Keywords: Pullback attractors; Trajectory attractors; Jeffreys–Oldroyd equations; Weak solutions; Existence theorems

1. Introduction

When mathematically describing the dynamics of a system, we identify possible states of the system with points of an abstract mathematical space called the phase space, and we prescribe an evolution law. In the simplest case of a deterministic autonomous system without memory the evolution law can be given in terms of a semigroup of evolutionary operators acting in the phase space. More sophisticated cases require the notion of a process (a biparametric family of operators) or trajectory spaces and families thereof. In this context the trajectory is a function $\mathbb{R}_+ \to E$, where $\mathbb{R}_+ = [0, +\infty)$ and E is the phase space. Each trajectory corresponds to a particular scenario of the evolution. If the dynamics of the system is described in terms of a

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semigroup of operators or a process, the trajectories can be defined by means of the evolutionary equations. If trajectory spaces are used, trajectories are postulated. Approaches involving trajectory spaces are the most general ones. In particular, trajectories make it possible to consider indeterministic dynamics, i.e. such that the initial state of the system may not uniquely determine the evolutions. For this reason trajectory and global attractors of trajectory spaces are very important in mathematical fluid mechanics, as in this domain the lack of uniqueness results is a commonplace.

There exists a class of systems such that a major part of potential states are unobservable in the sense that their existence is limited in time. The theory of dynamical systems introduces the notion of the attractor to deal with such systems. If a system has an attractor, initial data may become 'forgotten' under the evolution, and the limit regimes are due to intrinsic properties of the system. Mathematically, an attractor is characterised by the attraction property, which is usually accompanied by such requirements as minimality and compactness.

The notion of a trajectory attractor is the most important one in the theory of attractors of trajectory spaces. Trajectory attractors consist of functions $\mathbb{R}_+ \to E$ representing prototypic behaviour of trajectories. This means that the behaviour of any trajectory eventually resembles that of the functions belonging to the trajectory attractor.

The theory of trajectory attractors goes back to [1,2]. These papers feature the attractors of three-dimensional Navier–Stokes equations, which are notorious for its lack of uniqueness of weak solutions, and became a breakthrough in the analysis of indeterministic dynamics. The theory was further developed in [3–7], see also the reviews [8–10] and monographs [11–13].

Given an autonomous system with an attractor, it is only a matter of elapsed time, when the initial data gets 'forgotten'. In nonautonomous systems the absolute times of both start and check are to be taken into account. As a consequence, there is more than one way to generalise the notion of attractor to nonautonomous systems.

One well-established approach is to consider uniform attractors [3,5]. The uniformity of attraction is understood with respect to the initial time. Thus, given a bounded set D in the phase space, the trajectories starting in D are expected to land in a given neighbourhood of the attractor in a fixed time h_D no matter *when* they start. This resembles the attraction in autonomous system, as the absolute time of start and check is essentially irrelevant. For this reason uniform attractors are rather strong, i.e. they only exist in a rather narrow class of systems. In particular, when dealing with specific equation, one has to impose rather restricting assumptions on the time-dependent terms.

Another option is to use pullback attractors, which are less exigent. They were first considered in [14,15]. Initially, the theory of pullback attractors was naturally developed in the framework of processes (biparametric families of operators describing the evolution of nonautonomous systems). The infinite-dimensional setting of this theory has become quite rich both in abstract results and in applications, see e.g. the excellent monograph [16]. In particular, there are a number of results concerning pullback attractors of Newtonian fluids as well as certain non-Newtonian ones [17–23]. However, typical lack of uniqueness impedes the use of processes in fluid mechanics.

The notion of pullback attractor has recently been ported to trajectory spaces [7]. Even though the definitions of trajectory and global attractors are rather involved, the concept of pullback attraction is fairly intuitive. We sketch it here.

Considering a nonautonomous system, we start by defining a family of trajectory spaces. Given initial time $\xi \in \mathbb{R}$, consider the set $\mathcal{H}_\xi^+$ of trajectories starting at this time. The trajectories are parametrised not with absolute time, but with relative time elapsed from time ξ. Fix the time