Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 5075–5102
Journal of Differential Equations
www.elsevier.com/locate/jde

The behavior of the free boundary for reaction–diffusion equations with convection in an exterior domain with Neumann or Dirichlet boundary condition

Ross G. Pinsky

Department of Mathematics, Technion—Israel Institute of Technology, Haifa, 32000, Israel

Received 21 May 2015
Available online 18 December 2015

Abstract

Let

$$\mathcal{L} = A(r)\frac{d^2}{dr^2} - B(r)\frac{d}{dr}$$

be a second order elliptic operator and consider the reaction–diffusion equation with Neumann boundary condition,

$$\begin{aligned}&\mathcal{L}u = \Lambda u^p \text{ for } r \in (R, \infty);\\&u'(R) = -h;\\&u \geq 0 \text{ is minimal,}\end{aligned}$$

where $p \in (0, 1)$, $R > 0$, $h > 0$ and $\Lambda = \Lambda(r) > 0$. This equation is the radially symmetric case of an equation of the form

$$\begin{aligned}&Lu = \Lambda u^p \text{ in } \mathbb{R}^d - \bar{D};\\&\nabla u \cdot \bar{n} = -h \text{ on } \partial D;\end{aligned}$$

E-mail address: pinsky@math.technion.ac.il.
URL: http://www.math.technion.ac.il/~pinsky/.

http://dx.doi.org/10.1016/j.jde.2015.11.035

$$u \geq 0 \text{ is minimal,}$$

where

$$L = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}$$

is a second order elliptic operator, and where $d \geq 2$, $h > 0$ is continuous, $D \subset R^d$ is bounded, and $\bar{n}$ is the unit inward normal to the domain $\mathbb{R}^d - \bar{D}$. Consider also the same equations with the Neumann boundary condition replaced by the Dirichlet boundary condition; namely, $u(R) = h$ in the radial case and $u = h$ on ∂D in the general case. The solutions to the above equations may possess a free boundary. In the radially symmetric case, if $r^*(h) = \inf\{r > R : u(r) = 0\} < \infty$, we call this the radius of the free boundary; otherwise there is no free boundary. We normalize the diffusion coefficient A to be on unit order, consider the convection vector field B to be on order r^m, $m \in R$, pointing either inward $(-)$ or outward $(+)$, and consider the reaction coefficient Λ to be on order r^{-j}, $j \in R$. For both the Neumann boundary case and the Dirichlet boundary case, we show for which choices of m, $(\pm)$ and j a free boundary exists, and when it exists, we obtain its growth rate in h as a function of m, $(\pm)$ and j. These results are then used to study the free boundary in the non-radially symmetric case.

MSC: 35R35; 35J61

Keywords: Reaction–diffusion equation; Free boundary; Exterior domain; Neumann boundary condition; Boundary flux; Dirichlet boundary condition

1. Introduction and statement of results

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set with smooth boundary such that $\mathbb{R}^d - \bar{D}$ is connected. Let

$$L = \sum_{i,j=1}^{d} a_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} \tag{1.1}$$

be a strictly elliptic operator in $\mathbb{R}^d - D$ with smooth coefficients $a = \{a_{i,j}\}_{i,j=1}^d$ and $b = \{b_i\}_{i=1}^d$, and let $\Lambda > 0$ be a smooth function on $\mathbb{R}^d - D$.

In this paper, we consider two reaction–diffusion equations with sub-linear absorption, which are of the same form, but which have different boundary conditions, one the Neumann condition and the other the Dirichlet condition. We first describe the case of the Neumann boundary condition. We consider the following reaction–diffusion equation in the exterior domain $\mathbb{R}^d - \bar{D}$,

$$\begin{aligned} &Lu = \Lambda u^p \text{ in } \mathbb{R}^d - \bar{D};\\ &\nabla u \cdot \bar{n} = -h \text{ on } \partial D;\\ &u \geq 0 \text{ is minimal,} \end{aligned} \tag{1.2}$$