Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 260 (2016) 5378–5420
Journal of Differential Equations
www.elsevier.com/locate/jde

Decay estimates for the wave equation in two dimensions

Marius Beceanu

UC Berkeley Mathematics Department, Berkeley, CA 94720, United States

Received 6 June 2015; revised 22 November 2015

Available online 29 December 2015

Abstract

We establish Strichartz estimates (both reversed and some direct ones), pointwise decay estimates, and weighted decay estimates for the linear wave equation in dimension two with an almost scaling-critical potential, in the case when there is no resonance or eigenvalue at the edge of the spectrum.

We also prove some simple applications.

MSC: 35L05; 35B34; 35L71; 35B45

Keywords: Wave equation; Dimension two; Reversed Strichartz estimates; Pointwise decay estimates; Strichartz estimates

1. Introduction

1.1. Results

Consider the linear wave equation with a real-valued scalar potential in dimension two:

$$f_{tt} - \Delta f + Vf = F,\ f(0) = f_0,\ f_t(0) = f_1. \tag{1.1}$$

A natural condition for equation (1.1) to be well-posed is that $H = -\Delta + V$ should be self-adjoint. The existence of a self-adjoint extension was shown in [31] under the assumption that

E-mail address: mbeceanu@berkeley.edu.

http://dx.doi.org/10.1016/j.jde.2015.12.009
0022-0396/

$$\lim_{\epsilon \to 0} \sup_y \int_{|x-y|<\epsilon} |V(x)| \log_- |x-y| = 0.$$

If $H = -\Delta + V$ is self-adjoint, then the solution is given by

$$f(t) = \cos(t\sqrt{H}) f_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} f_1 + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} F(s)\, ds.$$

The following quantity, called energy, is constant as a function of t and remains bounded for all time if it is initially finite:

$$E[f](t) := \int_{\mathbb{R}^2 \times \{t\}} f_t^2 + |\nabla f|^2 + V f^2 \, dx.$$

Under rather general assumptions, the spectrum of the Hamiltonian H consists of the absolutely continuous component $[0, \infty)$ and the possibly empty point spectrum, containing negative energy eigenstates and zero energy eigenfunctions or resonances (see [20] and [25] concerning the absence of positive eigenvalues).

The solution's projection on the point spectrum of H lacks any decay and may even have exponential growth. Thus, in order to obtain dispersive estimates, we must first project away from the point spectrum. Zero energy states pose an even more serious obstruction: even after projecting them away, dispersion may only take place at a suboptimal rate or not at all. This is why we assume the absence of zero energy eigenfunctions and resonances in this paper.

Since the free ($V = 0$) linear equation (1.1) presents a resonance at energy zero in dimension two, some estimates obtained in the presence of a potential may be better than in the free case, as long as the potential eliminates the zero energy resonance. This is the case for inequalities (1.2), (1.4), (1.10), (1.11), (1.18), and (1.19) in this paper.

In the generic case when there is no resonance or eigenvalue at the edge of the spectrum of H, we establish several estimates for the projection on the continuous spectrum of the solution to equation (1.1). We prove pointwise $t^{-1/2}$ decay estimates, weighted integrable-in-time decay estimates, reversed Strichartz estimates, and some ordinary Strichartz estimates. All of them take place under almost scaling-invariant (hence optimal) decay conditions on the potential V.

One novelty of our results is that we prove reversed Strichartz estimates for the wave equation in dimension two, that is estimates that hold in the reversed Strichartz norms

$$\|f\|_{L^q_x L^r_t} := \left(\int_{\mathbb{R}^2} \|f(x, \cdot)\|_{L^r_t}^q \, dx \right)^{\frac{1}{q}}.$$

Such estimates have applications in the study of semilinear equations, see for example [5] or Proposition 1.6. Another application is proving the pointwise convergence of the solution of the wave equation to the initial data, see [6].

Although there is some overlap between the range of allowed exponents for reversed Strichartz estimates and the ones for ordinary Strichartz estimates, neither of them is contained in the