Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 260 (2016) 5510–5531

Journal of Differential Equations

www.elsevier.com/locate/jde

On the flow of non-axisymmetric perturbations of cylinders via surface diffusion

Jeremy LeCrone [a,*], Gieri Simonett [b]

[a] *Department of Mathematics, Kansas State University, Manhattan, KS, USA*
[b] *Department of Mathematics, Vanderbilt University, Nashville, TN, USA*

Received 2 June 2015

Available online 23 December 2015

Abstract

We study the surface diffusion flow acting on a class of general (non-axisymmetric) perturbations of cylinders $\mathcal{C}_r$ in $\mathbb{R}^3$. Using tools from parabolic theory on uniformly regular manifolds, and maximal regularity, we establish existence and uniqueness of solutions to surface diffusion flow starting from (spatially-unbounded) surfaces defined over $\mathcal{C}_r$ via scalar height functions which are uniformly bounded away from the central cylindrical axis. Additionally, we show that $\mathcal{C}_r$ is normally stable with respect to 2π-axially-periodic perturbations if the radius $r > 1$, and unstable if $0 < r < 1$. Stability is also shown to hold in settings with axial Neumann boundary conditions.

MSC: primary 35K93, 53C44; secondary 35B35, 35B32, 46T20

Keywords: Surface diffusion; Well posedness; Unbounded surfaces; Maximal regularity; Nonlinear stability; Implicit function theorem

* Corresponding author.
E-mail addresses: lecronjs@ksu.edu (J. LeCrone), gieri.simonett@vanderbilt.edu (G. Simonett).
URLs: http://www.math.ksu.edu/~lecronjs/ (J. LeCrone), http://www.vanderbilt.edu/math/people/simonett (G. Simonett).

http://dx.doi.org/10.1016/j.jde.2015.12.008

1. Introduction

The surface diffusion flow is a geometric evolution law in which the normal velocity of a moving surface equals the Laplace–Beltrami operator of the mean curvature. More precisely, we assume in the following that Γ_0 is a closed embedded surface in $\mathbb{R}^3$. Then the surface diffusion flow is governed by the law

$$V(t) = \Delta_{\Gamma(t)} H_{\Gamma(t)}\,, \qquad \Gamma(0) = \Gamma_0. \tag{1.1}$$

Here $\Gamma = \{\Gamma(t) : t \geq 0\}$ is a family of hypersurfaces, $V(t)$ denotes the velocity in the normal direction of Γ at time t, while $\Delta_{\Gamma(t)}$ and $H_{\Gamma(t)}$ stand for the Laplace–Beltrami operator and the mean curvature of $\Gamma(t)$ (the sum of the principal curvatures in our case), respectively. Both the normal velocity and the mean curvature depend on the local choice of the orientation. Here we consider the case where $\Gamma(t)$ is embedded and encloses a region $\Omega(t)$, and we then choose the outward orientation, so that V is positive if $\Omega(t)$ grows and $H_{\Gamma(t)}$ is positive if $\Gamma(t)$ is convex with respect to $\Omega(t)$.

The unknown quantity in (1.1) is the position and the geometry of the surface $\Gamma(t)$, which evolves in time. Hidden in the formulation of the evolution law (1.1) is a nonlinear partial differential equation of fourth order. This will become more apparent below, where the equation is stated more explicitly.

It is an interesting and significant fact that the surface diffusion flow evolves surfaces in such a way that the volume enclosed by $\Gamma(t)$ is preserved, while the surface area decreases (provided, of course, these quantities are finite). This follows from the well-known relationships

$$\frac{d}{dt}\mathrm{Vol(t)} = \int_{\Gamma(t)} V(t)\, d\sigma = \int_{\Gamma(t)} \Delta_{\Gamma(t)} H_{\Gamma(t)}\, d\sigma = 0 \tag{1.2}$$

and

$$\begin{aligned}\frac{d}{dt}\mathrm{A(t)} = \int_{\Gamma(t)} V(t) H_{\Gamma(t)}\, d\sigma &= \int_{\Gamma(t)} \left(\Delta_{\Gamma(t)} H_{\Gamma(t)}\right) H_{\Gamma(t)}\, d\sigma \\ &= -\int_{\Gamma(t)} \left|\mathrm{grad}_{\Gamma(t)} H_{\Gamma(t)}\right|^2_{\Gamma(t)}\, d\sigma \leq 0,\end{aligned} \tag{1.3}$$

where Vol(t) denotes the volume of $\Omega(t)$ and $A(t)$ the surface area of $\Gamma(t)$, respectively. Hence, the preferred ultimate states for the surface diffusion flow in the absence of geometric constraints are spheres. It is also interesting to note that the surface diffusion flow can be viewed as the H^{-1}-gradient flow of the area functional, a fact that was first observed in [20]. This particular structure has been exploited in [29,30] for devising numerical simulations.

The mathematical equations modeling surface diffusion go back to a paper by Mullins [32] from the 1950s, who was in turn motivated by earlier work of Herring [24]. Since then, the surface diffusion flow (1.1) has received wide attention in the mathematical community (and also by scientists in other fields), see for instance Asai [6], Baras, Duchon, and Robert [8], Bernoff, Bertozzi, and Witelski [9], Cahn and Taylor [11], Cahn, Elliott, and Novick-Cohen [10], Daví and Gurtin [14], Elliott and Garcke [17], Escher, Simonett, and Mayer [18], Escher and Mucha [19],