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# A moderate deviation principle for 2-D stochastic Navier–Stokes equations

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**Abstract**

In this paper, we prove a central limit theorem and establish a moderate deviation principle for two-dimensional stochastic Navier–Stokes equations with multiplicative noise. The weak convergence method plays an important role.


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*Keywords:* Stochastic Navier–Stokes equations; Central limit theorem; Moderate deviation principle

## 1. Introduction

Consider the two-dimensional stochastic Navier–Stokes equation with Dirichlet boundary condition, which describes the time evolution of an incompressible fluid,

$$\frac{\partial u^\varepsilon(t)}{\partial t} - \Delta u^\varepsilon(t) + \big(u^\varepsilon(t)\cdot\nabla\big)u^\varepsilon(t) + \nabla p(t,x) = f(t) + \sqrt{\varepsilon}\sigma\big(t, u^\varepsilon(t)\big)\frac{dW(t)}{dt}, \qquad (1.1)$$

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with the conditions

$$\begin{cases} (\nabla \cdot u^\varepsilon)(t,x) = 0, & \text{for } x \in D,\ t > 0, \\ u^\varepsilon(t,x) = 0, & \text{for } x \in \partial D,\ t \geq 0, \\ u^\varepsilon(0,x) = u_0(x), & \text{for } x \in D, \end{cases} \tag{1.2}$$

where $D$ is a bounded open domain of $\mathbb{R}^2$ with regular boundary $\partial D$, $u^\varepsilon(t,x) \in \mathbb{R}^2$ denotes the velocity field at time $t$ and position $x$, $p(t,x)$ denotes the pressure field, $f$ is a deterministic external force, and $W(\cdot)$ is a Wiener process.

To formulate the stochastic Navier–Stokes equation, we introduce the following standard spaces: let

$$V = \{v \in H_0^1(D;\mathbb{R}^2) : \nabla \cdot v = 0, \text{ a.e. in } D\},$$

with the norm

$$\|v\|_V := \left(\int_D |\nabla v|^2 dx\right)^{\frac{1}{2}} = \|v\|,$$

and let $H$ be the closure of $V$ in the $L^2$-norm

$$|v|_H := \left(\int_D |v|^2 dx\right)^{\frac{1}{2}} = |v|.$$

Define the operator $A$ (Stokes operator) in $H$ by the formula

$$Au := -P_H \Delta u, \quad \forall u \in H^2(D;\mathbb{R}^2) \cap V,$$

where the linear operator $P_H$ (Helmhotz–Hodge projection) is the projection operator from $L^2(D;\mathbb{R}^2)$ to $H$, and define the nonlinear operator $B$ by

$$B(u,v) := P_H\big((u \cdot \nabla)v\big),$$

with the notation $B(u) := B(u,u)$ for short.

By applying the operator $P_H$ to each term of (1.1), we can rewrite it in the following abstract form:

$$du^\varepsilon(t) + Au^\varepsilon(t)dt + B\big(u^\varepsilon(t)\big)dt = f(t)dt + \sqrt{\varepsilon}\sigma\big(t, u^\varepsilon(t)\big)dW(t), \tag{1.3}$$

with the initial condition $u^\varepsilon(0) = x$ for some fixed point $x$ in $H$.

As the parameter $\varepsilon$ tends to zero, the solution $u^\varepsilon$ of (1.3) will tend to the solution of the following deterministic Navier–Stokes equation

$$du^0(t) + Au^0(t)dt + B\big(u^0(t)\big)dt = f(t)dt, \quad \text{with } u^0(0) = x. \tag{1.4}$$