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Existence, uniqueness, analyticity, and Borel summability for Boussinesq equations

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Abstract

Through Borel summation methods, we analyze the Boussinesq equations for coupled fluid velocity and temperature fields:

$$u_t - \nu \Delta u = -P[u \cdot \nabla u - a e_2 \Theta] + f$$
$$\Theta_t - \mu \Delta \Theta = -u \cdot \nabla \Theta. \tag{1}$$

We prove that an equivalent system of integral equations in the Borel variable $p \in \mathbb{R}^+$ dual to $1/t$ has a unique solution in a class of exponentially bounded functions, implying the existence of a classical solution to (1) in a complex t-region that includes a real positive time axis segment. For analytic initial data and forcing, it is shown that the solution is Borel summable, implying that formal series in powers of t is Gevrey-1 asymptotic, and within the time interval of existence, the solution remains analytic with the same analyticity strip width as the initial data and forcing. We also determine conditions on the integral equation solution that improve the estimate for existence time.

Keywords: Boussinesq equation; Borel summability

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1. Introduction

We consider the Boussinesq equations for coupled fluid velocity and temperature fields derived under the assumption that the temperature induced density has negligible effect on momentum but causes a significant buoyant force. The corresponding evolution equations for $u : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d$ and $\Theta : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}$ for dimension $d = 2, 3$ in non-dimensional form are:

$$\begin{aligned} u_t - \nu \Delta u &= -P[u \cdot \nabla u - a e_2 \Theta] + f, \quad u(x,0) = u_0(x) \\ \Theta_t - \mu \Delta \Theta &= -u \cdot \nabla \Theta, \quad \Theta(x,0) = \Theta_0(x) \end{aligned} \tag{2}$$

where $P = I - \nabla \Delta^{-1}(\nabla \cdot)$ is the Hodge projection operator to the space of divergence free vector fields, e_2 is the unit vector aligned opposite to gravity, the parameter a is proportional to gravity, and (u, Θ) are the nondimensional fluid velocity and temperature fields. We assume the initial conditions u_0 and the forcing f are divergence free and, for the sake of simplicity, assume f to be time independent, although time dependence with some restrictions can be accommodated in a similar framework. Using standard energy methods, see for instance [22], existence of Leray type solutions in $L^\infty(0, T, L^2(\mathbb{R}^d)) \cap L^2(0, T, H^1(\mathbb{R}^d))$ follows easily for any $T > 0$. In $\mathbb{R}^2$ a unique classical global solution can be shown to exist for all time In [4], local existence and uniqueness for Boussinesq equation are shown in $L^p(0, T, L^q(\mathbb{R}^d))$ for $d < p < \infty$ and $\frac{d}{p} + \frac{2}{q} \le 1$. In $\mathbb{R}^3$ there is a unique solution under the additional assumption that the solution lies in $L^\infty(0, T, H^1(\mathbb{R}^3))$, see [4]. The case where $\mu = 0$ has also been considered in the literature, and global well-posedness is proved in [17] for $2 - d$.

In the problem above, the existence of classical solutions, globally in time, remains an open problem as it is for the limiting ($a \to 0$) Navier Stokes equation (NSE) in 3-D. Control of a higher order energy norm (like the H^1 norm of velocity) has remained a serious impediment despite extensive study of NSE. This motivates one to look for alternate formulations of existence that do not rely on energy bounds.

The primary purpose of this paper is to show that the Borel based methods, developed earlier in [10] and [13] in the context of Navier–Stokes equation, can be extended to other evolutionary PDEs (partial differential equations) such as the Boussinesq equation. This provides an alternate existence and uniqueness theory for a class of nonlinear PDEs. In this formulation, the question of global existence of solution to the PDE becomes one of asymptotics for known solution to the associated nonlinear integral equations. While the asymptotics are still difficult, it is interesting to note that an accelerated representation [13] (see (5) in the ensuing) for the related NSE results in a positive limiting kernel as $n \to \infty$, where majorization may be possible in terms of solution to a simpler integral equation. We also show (Theorem 2.3) here how information about solution to the integral equation on a finite interval in the dual variable for specific initial condition and forcing may be used to obtain better exponential bounds in the Borel plane implying a longer existence time for classical solutions to the associated PDEs.

Borel summability has been an active area of research. A vast literature has emerged recently in Borel summability theory, starting with the fundamental contributions of Ecalle (see e.g. [14] and [15]) whose consequences are far from being fully explored, and it is impossible to give a quick account of the breadth of this field (see for example [5] for more references). There has also been work in characterizing all small solutions for a generic system of ODEs [6] or difference equations [3]. There has been work on PDEs as well, starting with linear equations [19,2] followed by general results for a class of nonlinear system of PDEs in complex