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# The Boltzmann equation with frictional force for soft potentials in the whole space

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**Abstract**

This paper is concerned with the Cauchy problem of the Boltzmann equation with frictional force for both cutoff and non-cutoff soft potentials in the whole space and our main purpose is to establish its global solvability result near a given global Maxwellian and to deduce the temporal convergence rates of such a global solution toward the global Maxwellian when initial perturbation is sufficiently small. The analysis is based on the time-weighted energy method building also upon the recent studies of the cutoff Vlasov–Poisson–Boltzmann system [14,15], the non-cutoff Vlasov–Poisson–Boltzmann system [8], and non-cutoff Vlasov–Maxwell–Boltzmann system [9].


## 1. Introduction and main results

### 1.1. The problem

This paper is concerned with the following Boltzmann equation with external force proportional to the macroscopic velocity $u(t,x)$ in the whole space $\mathbb{R}^3$:

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$$\begin{aligned}
&\partial_t F + v \cdot \nabla_x F - \zeta u \cdot \nabla_v F = Q(F, F), \\
&u = \Bigg(\int\limits_{\mathbb{R}^3} vF dv\Bigg)\Bigg(\int\limits_{\mathbb{R}^3} F dv\Bigg)^{-1}
\end{aligned} \tag{1.1}$$

with initial data

$$F(0, x, v) = F_0(x, v). \tag{1.2}$$

Here $F = F(t, x, v) \geq 0$ stands for the velocity distribution functions for the particles with position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ at time $t \geq 0$ and the term $\zeta u$ represents the frictional force which is proportional to the macroscopic velocity $u(t, x)$ and we can normalize the positive constant $\zeta$ to be 1 without loss of generality. The bilinear collision operator $Q(F, G)$ acting only on the velocity variable is defined by

$$Q(F, G)(v) = \int\limits_{\mathbb{R}^3 \times \mathbb{S}^2} \mathbf{B}(v - v_*, \sigma)\big\{F\big(v'_*\big)G\big(v'\big) - F(v_*)G(v)\big\} d\sigma dv_*,$$

where in terms of velocities $v_*$ and $v$ before the collision, velocities $v'$ and $v'_*$ after the collision are defined by

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad u' = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

which follow from the conservation of momentum and kinetic energy during the collision process.

The Boltzmann collision kernel $\mathbf{B}(v - v_*, \sigma)$ depends only on the relative velocity $|v - v_*|$ and on the deviation angle $\theta$ given by $\cos\theta = \langle \sigma, (v - v_*)/|v - v_*|\rangle$, where $\langle \cdot, \cdot \rangle$ is the usual dot product in $\mathbb{R}^3$. As in [1–3,21], without loss of generality, we suppose that $\mathbf{B}(v - v_*, \sigma)$ is supported on $\cos\theta \geq 0$. Throughout the paper, the collision kernel $\mathbf{B}(v - v_*, \sigma)$ is further supposed to satisfy the following assumptions:

(A1) $\mathbf{B}(v - v_*, \sigma)$ takes the product form in its argument as

$$\mathbf{B}(v - v_*, \sigma) = \Phi\big(|v - v_*|\big)\mathbf{b}(\cos\theta)$$

with the kinetic part $\Phi$ and the angular part $\mathbf{b}$ being non-negative functions.

(A2) The angular part $\mathbf{b}(\cos\theta)$ is assumed to satisfy one of the following conditions:
- $\mathbf{b}(\cos\theta)$ satisfies Grad's angular cutoff assumption $0 \leq \mathbf{b}(\cos\theta) \leq C|\cos\theta|$;
- for the non-cutoff case, the angular function $\sigma \to \mathbf{b}(\langle \sigma, (v - v_*)/|v - v_*|\rangle)$ is not integrable on $\mathbb{S}^2$, i.e.

$$\int\limits_{\mathbb{S}^2} \mathbf{b}(\cos\theta) d\sigma = 2\pi \int\limits_0^{\pi/2} \sin\theta \mathbf{b}(\cos\theta) d\theta = \infty$$