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Non-negative global weak solutions for a degenerated parabolic system approximating the two-phase Stokes problem

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Abstract

We establish the existence of non-negative global weak solutions for a strongly coupled degenerated parabolic system which was obtained as an approximation of the two-phase Stokes problem driven solely by capillary forces. Moreover, the system under consideration may be viewed as a two-phase generalization of the classical Thin Film equation.

MSC: 35K41; 35K55; 35K65; 35Q35; 76A20

Keywords: Thin Film; Degenerated parabolic system; Non-negative global weak solutions

1. Introduction and the main result

In this paper we study the following system of one-dimensional degenerated parabolic equations

$$\begin{cases} \partial_t f = -\partial_x\left(f^3\partial_x^3 f + \dfrac{R}{2}\left(2f^3+3f^2g\right)\partial_x^3(f+g)\right), \\ \partial_t g = -\partial_x\left(\dfrac{3}{2}f^2g\partial_x^3 f + \dfrac{R}{2}\left(2\mu g^3+3f^2g+6fg^2\right)\partial_x^3(f+g)\right) \end{cases} \tag{1.1a}$$

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for $(t, x) \in (0, \infty) \times \mathcal{I}$, where $\mathcal{I} := (0, L)$ for some $L > 0$. This system models the motion of two immiscible thin fluid layers of the heights f and g, respectively. The layer of height f is located on an impermeable horizontal bottom, identified with the line $y = 0$, and the layer of height g is located on top of the first one. The system (1.1a) has been recently derived in [10] as a thin film approximation of the two-phase Stokes problem when the capillarity is the sole driving mechanism. The constants R and μ, which are both assumed to be positive, are determined by material properties of the fluids and are given by

$$\mu := \frac{\mu_f}{\mu_g} \quad \text{and} \quad R := \frac{\gamma_g}{\gamma_f},$$

with μ_f [resp. μ_g] denoting the viscosity coefficient of the fluid layer of height f [resp. g]. Moreover, γ_f [resp. γ_g] is the surface tension coefficient at the interface $y = f(t, x)$ [resp. $y = f(t, x) + g(t, x)$]. The system (1.1a) is supplemented by the initial conditions

$$f(0) = f_0, \qquad g(0) = g_0 \quad \text{in } \mathcal{I}, \tag{1.1b}$$

whereby f_0 and g_0 are assumed to be known non-negative functions, and by no-flux boundary conditions

$$\partial_x f = \partial_x g = \partial_x^3 f = \partial_x^3 g = 0, \quad x = 0, L. \tag{1.1c}$$

Let us first observe that if one of the fluid layers has constant zero height, then the system (1.1a) becomes, up to a scaling factor, the well-known Thin Film equation

$$\partial_t h = \partial_x\big(h^m \partial_x^3 h\big), \quad m > 0, \tag{1.2}$$

with $m = 3$. The theory of existence of weak solutions for the Thin Film equation (1.2) is well-established nowadays, cf. [1–5,16], to mention just some of the most important contributions. Recently, a theory for classical solutions of (1.2) was developed, cf. [13,18], Eq. (1.2) being defined only on the set of positivity of h which is allowed to evolve in time. We emphasize that it has been rigorously proved in [17,22] (see also [14]) that suitably rescaled solutions of the Stokes and of the Hele-Shaw problem converge towards corresponding solutions of Eq. (1.2) with $m = 3$ and $m = 1$, respectively. This issue is still an open problem in the context of (1.1a). Compared to the Thin Film equation (1.2), the system (1.1) is much more complex because it exhibits a strong coupling as both equations contain highest order derivatives of all the unknowns. Additionally, there are two sources of degeneracy because both interfaces may vanish on subsets of the interval $\mathcal{I}$.

It is worth mentioning that there exists also a two-phase generalization corresponding to the Thin Film equation (1.2) with $m = 1$, which has been derived in [9] for flows with capillary and gravity effects. The resulting system, which has been investigated in [11,20,21] in the presence of capillary and in [8,19] for flows driven only by gravity, appears as the thin layer approximation of the two-phase Muskat problem. Compared to (1.1a), the parabolic system obtained in [9] has much more structure: there are two energy functionals available and, furthermore, the system can be interpreted as a gradient flow for the L_2-Wasserstein distance in the space of probability measures with finite second moment. There are not many systems of equations which enjoy this nice geometric property. We mention here only the parabolic–parabolic Keller–Segel system